國立清華大學 101 學年度碩士班考試入學試題

系所班組別：數學系純粹數學組

考試科目（代碼）：代數與線性代數（0102）　　共 1 頁，第 1 頁　　*請在【答案卷】作答

Algebra and Linear Algebra

In the problems below, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{Z}_m$ denotes the additive group of integers mod $m > 0$ (if $m = p$ is a prime then $\mathbb{Z}_p$ is a field) and $\mathbb{Q}$ (resp. $\mathbb{C}$) denotes the field of rational numbers (resp. complex numbers).

(21%) 1. Let $A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 1 & -2 & -5 & 2 & 3 \\ -2 & 1 & 4 & -1 & 3 \\ -3 & 1 & 5 & -1 & 6 \end{pmatrix}$.

(12%) (a) Find the rank of A.

(9%) (b) Find a 5×5 matrix M with rank 2 such that $AM = O$, where O is the 4×5 zero matrix.

(22%) 2. Let $B = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

(9%) (a) Find the characteristic polynomial of B.

(13%) (b) Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}BP = D$.

(16%) 3. Let p be an odd prime. Consider the set G consisting of all the 3×3 matrices $A = (A_{ij})$ such that **(i)** A_{ij} lies in the field $\mathbb{Z}_p$ all i, j **(ii)** $A_{ij} = 0$ for $i > j$ and **(iii)** $A_{ii} \in \mathbb{Z}_p - \{0\}, 1 \le i \le 3$. G is a group under the operation of matrix multiplication with identity matrix $I_3 = (\delta_{ij})$ as the identity (you don't have to prove this).

(6%) (a) What is the order of the group G?

(10%) (b) Prove that G has only one Sylow p-subgroup.

(A Sylow p-subgroup of a finite group K is a subgroup which has order the maximum powers of p dividing the order of K.)

(20%) 4 Let $< 1 + 2\sqrt{-5} >$ be the principal ideal of the integral domain $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$ generated by $1 + 2\sqrt{-5}$.

(10%) (a) Show that the quotient ring $\mathbb{Z}[\sqrt{-5}] \ / < 1 + 2\sqrt{-5} >$ is a finite ring, that is, it has only a finite number of elements.

(10%) (b) Show that $< 1 + 2\sqrt{-5} >$ is not a prime ideal in $\mathbb{Z}[\sqrt{-5}]$.

(21%) 5 Let α, β be elements in $\mathbb{C}$ which are algebraic over $\mathbb{Q}$. Prove the following.

(11%) (a) $[\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\beta)(\alpha) : \mathbb{Q}(\beta)] \le [\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta)]$.

(10%) (b) If $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}][\mathbb{Q}(\beta) : \mathbb{Q}]$ then $\mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) = \mathbb{Q}$.