科目：工程數學B(3005)

校系所組：中央大學通訊工程學系（甲組、乙組）
交通大學電子研究所(甲組、乙A組、乙B組)
交通大學電控工程研究所
交通大學電機工程學系（甲組、乙組）
交通大學電信工程研究所（甲組）
交通大學生醫工程研究所(乙組)
清華大學電機工程學系(乙組、丙組)
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1. **(12%)** Let $\mathsf{V} = \{(a_1, a_2, a_3) : a_1, a_2, a_3 \in \mathbb{R}\}$ be a vector space over $\mathbb{R}$ with vector addition and scalar multiplication defined such that, for (a_1, a_2, a_3), $(b_1, b_2, b_3) \in \mathsf{V}$ and $c \in \mathbb{R}$,

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1 + \alpha, a_2 + b_2 + \alpha, a_3 + b_3 + \alpha)$$

and

$$c(a_1, a_2, a_3) = (ca_1 + \alpha(c-1), ca_2 + \alpha(c-1), ca_3 + \alpha(c-1)),$$

where α is some positive constant in $\mathbb{R}$.

(a) (4%) Find the zero vector in V, denoted by 0_{V}, such that $u + 0_{\mathsf{V}} = u$, for all $u \in \mathsf{V}$.

(b) (8%) Show that $\{(1,0,0), (0,1,0), (0,0,1)\}$ forms a basis for V.

2. **(13%)** Let $\mathsf{P}_3(\mathbb{R})$ be the vector space consisting of real polynomials with degree less than or equal to 3 and let $\mathsf{T} : \mathsf{P}_3(\mathbb{R}) \to \mathsf{P}_3(\mathbb{R})$ be a linear operator on $\mathsf{P}_3(\mathbb{R})$ defined such that

$$\mathsf{T}(f(x)) = -xf'(x) + 2f(x) + f(-1)x, \quad \forall f(x) \in \mathsf{P}_3(\mathbb{R}).$$

(a) (5%) Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1+x, 1+x^2, 1+x^3, x+x^2\}$ be ordered bases for $\mathsf{P}_3(\mathbb{R})$. Find matrix Q such that

$$[\mathsf{T}]_\alpha = QA,$$

where $[\mathsf{T}]_\beta$ is the matrix representation of T in the ordered basis β and $A \triangleq [\mathsf{I}_{\mathsf{P}_3(\mathbb{R})}]_\alpha^\beta$ is the matrix representation of the identity mapping in the ordered bases β and α.

(b) (8%) Let $\mathsf{S} = \{f(x) \in \mathsf{P}_3(\mathbb{R}) : 3f(1) - f(2) = 0\}$ be a subspace of $\mathsf{P}_3(\mathbb{R})$. Find a basis for the subspace

$$\mathsf{W} = \{\mathsf{T}(f(x)) : f(x) \in \mathsf{S}\}.$$

(Please show your derivations.)

3. **(15%)** Let $\mathsf{V} = \mathsf{P}_1(\mathbb{R})$ be the vector space consisting of all polynomials with real-valued entries and having degree less than or equal to one. Let T be a linear operator on V defined by $\mathsf{T}(f(x)) = 3f'(x) + f(2)x$. Determine the eigenvalues of T and find an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix, where $[\mathsf{T}]_\beta$ is the matrix representation of T with respect to the ordered basis β.

注意：背面有試題

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4. (10%) Let V = $\mathbb{R}^3$ with the inner product $\langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle = x_1x_2 + y_1y_2 + z_1z_2$ for $(x_1, y_1, z_1) \in$ V and $(x_2, y_2, z_2) \in V$, $v = (2, 1, 3)$, and W = $\{(x, y, z) : x + 3y - 2z = 0\}$ be a subspace of V. Find the orthogonal projection u of v on W.

5. (7%) [Function of a Random Variable] Let X be a geometrically distributed random variable with parameter $0 < p < 1$, i.e., $P(X = k) = (1-p)^{k-1}p$ for $k = 1, 2, \ldots$. Let $Y = \min(X, 10)$, i.e., $Y(\omega)$ is the minimum of $X(\omega)$ and 10 for each outcome ω. Please find the cumulative distribution function $F_Y(y)$ of Y.

6. (8%) [Roll a Die] Is the event that an even number shows up independent of the event that a multiple of three shows up?

7. (10%) [Independent Trials] Independent trials of flipping a fair coin are performed. Please calculate the probability $P(E)$ of the event E that the pattern HT appears before the pattern THH.

8. (10%) Let X be a nonnegative continuous random variable, and assume that its probability density function is $f_X(x) = C\cos^2(\pi x)$, $-\frac{1}{2} < x < \frac{1}{2}$ and $f_X(x) = 0$, otherwise.

(a) (3%) What should the constant C be?

(b) (7%) Find a transform h, such that $Y = h(X)$ is a uniform random variable in $(-\frac{1}{2}, \frac{1}{2})$.

9. (15%) Let (X, Y) be jointly continuous random variables and their joint probability density function is $f(x, y) = Ce^{-a|x|-b|y|}$, $-\infty < x, y < \infty$, where $a, b > 0$ are arbitrary constants.

(a) (3%) Find C, such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$.

(b) (5%) Are X and Y independent? Why or why not?

(c) (7%) Let $Z = a|X| + b|Y|$. Find out its probability density function $f_Z(t)$.