國　立　清　華　大　學　命　題　紙

八十五學年度 動力機械 系（所） 甲乙丙丁 組碩士班研究生入學考試

科目 工程數學 科號 2903 2603 2703 2803 共 2 頁第 1 頁 *請在試卷【答案卷】內作答

QUESTION 1 (10%)

(a) Show that let $x = \exp(z)$ then $x \dfrac{dy}{dx} = \dfrac{dy}{dz}$ and $x^2 \dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2}$.

(b) Solve the Euler equation

$$x^2\, y'' + 2\, x\, y' - 12\, y = \sqrt{x} \qquad x > 0$$

by using the method of transformation that you have verified in part (a).

QUESTION 2 (20%)

(a) Let C be a closed curve and Ω be the region enclosed by C on xy-plan. Use Green's theorem to verify that the area of Ω can be evaluated from

$$\iint_\Omega dA = \oint_C x\, dy = \oint_C (-y)\, dx = \frac{1}{2} \oint_C (-y)\, dx + x\, dy$$

(b) Use the formula in part (a) to show that the area enclosed by the ellipse $[(x^2/a^2) + (y^2/b^2) = 1]$ is $(\pi\, a\, b)$.

[Hint]: The ellipse can be written as

$x = a\ (\cos t)$ and $y = b\ (\sin t)$ for $0 \le t \le 2\pi$.

QUESTION 3 (20%)

(a) The Legendre Polynomials $P_n(x)$ form an orthogonal set in the interval $[-1, 1]$ such that any piecewise continuous function $f(x)$ is expressible as

$$f(x) = \sum_{n=0}^{\infty} c_n\, P_n(x)$$

Show that the coefficients of the Fourier-Legendre series are

$$c_n = \frac{1}{\|P_n\|^2} \int_{-1}^{1} f(x)\, P_n(x)\, dx$$

(b) Find c_n, $n = 0, 1, 2 \cdots \infty$ for $f(x) = 6\, x^2 - 2\, x + 3$

[Hint]: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3\, x^2 - 1)$,

$P_3(x) = \frac{1}{2}(5\, x^3 - 3\, x)$, $P_4(x) = \frac{1}{8}(35\, x^4 - 30\, x^2 + 3)$

$$\|P_n\|^2 = \frac{2}{2n + 1}$$

QUESTION 4

The Laplace transform of a function f(t) is

$$L\{f(t)\} = \int_0^\infty f(t)e^{-pt}dt.$$

(a)Show that

$$L\{f'(t)\} = pL\{f(t)\} - f(0^+).$$

(3%)

(b)What is $L\{te^{-t}\}$? (3%)

(c)What is the solution of the differential equation

$$y' + y = te^{-t}, \quad y(0) = 0$$

(4%)

QUESTION 5

(a) If

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

what is A^n ? (5%)

(b) Let

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

, find the eigenvalues and eigenvectors of B. (10%)

(c) Are the eigenvectors of B linearly independent? (5%)

QUESTION 6

(a) What is the residue of $\tan z$ at $z = \frac{\pi}{2}$? (5%)

(b) Find the value of $\int_C \frac{5z-\text{[illegible]}}{z(z-1)(z-2)}dz$, where C is a closed curve lying in the annulus $1 < |z+1| < 2$. (6%)

(c) If f(z) is an analytic function, show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

(9%)