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A generalised equivalent neighbour model: I. Derivation of the high-temperature lattice constants†

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Abstract. A new equivalent-neighbour model is introduced in which the spins within a hypercube of volume $(2R + 1)^d$ interact equally with the spin at the centre of the hypercube. The lattice dimensionality d and the range of the interaction R are variables. The concept of the coincidable occurrence factors is introduced. Connections between the coincidable occurrence factors and the high-temperature lattice constants are shown. A systematic way of deriving the high-temperature lattice constants through the use of the coincidable occurrence factors is developed. The high-temperature lattice constants are derived up to eighth order. The extension of d and R from integers to real values is discussed. The present model can also be extended to an anisotropic one in which the ranges of interactions are different in different lattice directions.

1. Introduction

Critical properties of a magnetic system depend not only on the form of interactions between pairs of spins, but also on the feature of the lattice on which the spins are located. The dependence of critical behaviour on the lattice, such as the dimensionality d, the range of the interaction R and the lattice anisotropy, has been an area of particular interest for the past decade. Exact solutions are known only for a few models on two-dimensional lattices (Domb and Green 1972). For the cases that the range of the interaction $R \to \infty$ (Stanley 1971, Tanaka and Mannari 1976), and that the dimensionality $d \geqslant 4$ (Wilson and Kogut 1974), the critical exponents are known to be equal to the mean-field results. When $d = 4 - \epsilon$, exact ϵ expansions of the critical exponents are obtained from the renormalisation group approach (Wilson and Kogut 1974). For general lattices, critical properties are investigated by various approximate methods. In particular the series expansion method has provided much significant information (Domb and Green 1974).

In the series expansion method, various spin Hamiltonians on regular two- and three-dimensional lattices with nearest-neighbour (NN) interactions have been extensively studied. Since the spins may interact with next-nearest-neighbour (NNN) spins and with spins at a longer distance, a model with the interaction constant J_1 between NN spins and J_2 between NNN spins was studied for the Ising model by Domb and Potts (1951) and Dalton and Wood (1969); for the Heisenberg model by Wojtowicz and Joseph (1964), and Dalton and Wood (1965); and for the n vector model by Paul and Stanley

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(1972a). When further neighbours are taken into account the complication in the investigation increases seriously. Two simplified models were developed: (i) a long range force of the form $J(r) = r^{-(d+\sigma)}$ (Joyce 1966); and (ii) $J(r) = J$ for $r \leqslant R$, and vanishes otherwise.

The second approach was first introduced by Domb and Dalton (1966), and is called the equivalent-neighbour model. High-temperature lattice constants were calculated for a variety of two- and three-dimensional lattices in which the interactions are extended to three shells (third-nearest neighbours). Longer high-temperature series expansions for the two-shell equivalent-neighbour model were later calculated by Bowers and Woolf (1969). The low-temperature series expansions were derived by Dalton and Wood (1969). In the equivalent-neighbour model, the complication in the derivations of lattice constants increases rapidly as the number of shells r increases and evaluations of the lattice constants become formidable for $r > 4$.

One of the purposes of the present paper is to introduce a new equivalent-neighbour model for which the lattice constants as functions of the range of the interaction R can be obtained and R can be extended from integers to real values.

To understand the dimensional dependence of the critical properties, Fisher and Gaunt (1964) first introduced a d-dimensional hypercubical lattice. Critical properties of the hypercubical lattice for general integral values of d were subsequently studied by many authors (Stanley 1969, Baker 1974, Gerber and Fisher 1974). The hypercubical lattice is a loose-packed lattice with coordination number $q = 2d$. Van Dyke and Camp (1976) then introduced a d-dimensional hypertriangular lattice in which any pairs of the d lattice translation vectors are the lattice translation vectors of a two-dimensional triangular lattice. It is a close-packed lattice with $q = d(d + 1)$. Van Dyke and Camp investigated the high-temperature susceptibility series for the Ising model for general values of d. Details of their work have not been reported. The new equivalent-neighbour model that we will introduce is a lattice in which the dimensionality d and the coordination number q are arbitrary values.

To investigate the dependence of critical properties on the lattice anisotropy, Oitmaa and Enting (1971, 1972) have developed high-temperature series expansions for the Ising model on a simple cubic lattice with exchange constants J in the x and y directions, and ηJ in the z direction. Similar studies have been carried out by Rapaport (1971) and by Paul and Stanley (1971, 1972b) for the simple cubic, and the face-centered cubic lattices. Subsequently there have been extensive studies on dimensional crossover for systems with different exchange constants in different lattice directions. In this paper we introduce a new kind of lattice anisotropy.

It is the ranges of interactions, instead of the strengths of interactions, that are different in different lattice directions. For instance, the range of the interaction is R_1 in the z direction and is R_2 in the x and y directions.

In the following sections we first define the new equivalent-neighbour model. The method for deriving the high-temperature lattice constants as functions of d and R are then described. The high-temperature lattice constants are given to the eighth order. Applications of the present lattice will be reported in subsequent papers.

2. The generalised equivalent-neighbour model

We will consider a system of N spins located on a d-dimensional lattice of volume $V = N$, i.e., the density of spins $N/V = 1$. Each spin within a hypercube of volume

$(2R + 1)^d$ interacts equally with the spin located at the centre of the volume. That is, the exchange constant

$$J(\boldsymbol{r}_i - \boldsymbol{r}_j) = J \text{ if } 0 < \max\{|x_{i1} - x_{j1}|, |x_{i2} - x_{j2}|, \ldots\} \leqslant R,$$
$$= 0 \text{ otherwise} \qquad (1)$$

where $\boldsymbol{r}_i = (x_{i1}, x_{i2}, x_{i3}, \ldots)$. The difference between the present model and the equivalent model introduced previously by Domb and Dalton (1966, to be referred to as the DD model) is that they defined

$$J(\boldsymbol{r}_i - \boldsymbol{r}_j) = J \text{ if } 0 < |\boldsymbol{r}_i - \boldsymbol{r}_j|^2 \leqslant r,$$
$$= 0 \text{ otherwise.} \qquad (2)$$

In other words, the range of the interaction is hyperspherically symmetric in the DD model while it is hypercubically symmetric in the present model. From another point of view, the Euclidean metric is defined for the interaction range in the DD model while the uniform metric is defined for the present model.

Equation (1) only defines the exchange constant. Our model lattice is not uniquely defined unless the lattice structure is specified. In our model the dimensionality d is to be extended to non-integral values. It is impossible, however, to define the structure (i.e., to specify the lattice translation vectors) of a lattice of non-integral dimension. Since the quantities to be calculated for a lattice in the series expansion method are the lattice constants, we can obtain the lattice constants for a lattice of general dimension in two different ways:

(i) We ignore the lattice structure and assume that the spins are distributed uniformly in the lattice such that there are $(L)^d$ spins in each d-dimensional hypercube of any length L.

(ii) We specify the lattice structure, such as the hypercubical, or the hypertriangular lattice for integral values of d, and derive the general expressions of the lattice constants as functions of d and R. The variables d and R are then extended from integers to real numbers.

In the present paper we shall adopt the second approach, and assume that the spins are located at the lattice sites of a simple hypercubical lattice. The reasons are:

(i) It is much easier to derive the lattice constants for the hypercubical lattice. As will be seen in the following sections, in deriving the lattice constants, each lattice dimension can be treated independently. Moreover, an important part of the the calculation can be done by computer.

(ii) In some special cases the present model is identical to the DD model. For $d = 3$, the present model with $R = 1$ is identical to the simple cubic lattice with $r = 3$ in the DD model. As $d = 2$ the present model with $R = 1$ and 2 are respectively identical to the simple quadratic lattice with $r = 2$ and 5 in the DD model. For $d = 1$, these two models are the same for $r = R =$ integers.

(iii) If the range of the interaction is R_1 in d_1 lattice directions and is R_2 in the other d_2 directions, the system reduces from a $(d_1 + d_2)$-dimensional lattice to a d_2-dimensional lattice as R_1 approaches zero. This anisotropic model can be applied to the analysis of the dimensional crossover. If approach (i) is used, the coordination number q vanishes as $R_1 \to 0$.

3. Direct calculation of the lattice constants

Consider a spin, labelled i, located at the origin as shown in figure 1. If the range of the interaction is R, the neighbouring spins that interact with spin i are those enclosed in the box of volume $(2R+1)^d$. The generalised coordination number of the lattice, i.e. the number of spins interacting with a given spin on the lattice, is $q=(2R+1)^d-1$.

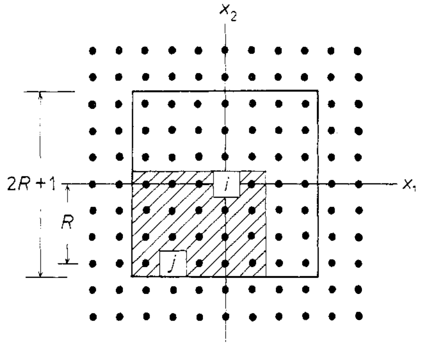


Figure 1. If the spin i is located at the origin the spins within the box of volume $(2R+1)^d$ will interact with the spin i. With the spin j located at $(x_1, x_2, \ldots)$ the spins within the shaded region of volume $(2R+1-|x_1|)(2R+1-|x_2|)\ldots$ will interact with spins i and j. In this figure $R=3$, $x_1=-2$, $x_2=-3$.

To calculate the high-temperature lattice constant p_3, let the three spins that interact with one another be labelled i, j and k. If the spin i is the one at the origin, the spin j can be any one within the box shown in figure 1. If the spin j is located at the position shown, then the third spin k can be any one within the shaded region shown in figure 1. That is, with the spin i at the origin and spin j at $(x_1, x_2, \ldots, x_d)$, the number of ways of selecting the spin k is

$$-2+(2R+1-|x_1|)(2R+1-|x_2|)\ldots(2R+1-|x_d|). \tag{3}$$

The second term in (3) is nothing but the volume of the shaded region and the term -2 indicates that neither the spin i nor the spin j should be chosen as the third spin. We have

$$6p_3=\sum_{x_1}\sum_{x_2}\ldots\sum_{x_d}[-2+(2R+1-|x_1|)(2R+1-|x_2|)\ldots], \tag{4}$$

where the summations are over all integers $0, \pm1, \pm2, \ldots \pm R$, except $x_1=x_2=x_3\ldots=0$. The factor 6 on the left hand side is the symmetry number of the triangle. We multiply the high-temperature lattice constant by the symmetry number because in the right hand side of equation (4) the vertices (spins) of a graph (cluster of interacting spins) are labelled such that the number of different ways of labelling the graph is equal to the symmetry number of the graph (for definition see p 7 of Domb and Green 1974).

It is straightforward to perform the summation. We obtain

$$6p_3=(3R^2+3R+1)^d-3(2R+1)^d+2 \tag{5a}$$

$$=(3R^2+3R+1)^d-3q-1. \tag{5b}$$

Similarly we can calculate p_4, p_{5d}, etc. The complexity in the calculation of p_{nx} increases very rapidly as n increases, and p_{nx} can hardly be calculated directly. In the following section we will describe a method to derive the high-temperature lattice constants systematically.

4. A systematic method of deriving the high-temperature lattice constants

4.1. Coincidable occurrence factors

The first term on the right hand side of equation (5*b*) would be the lattice constants $6p_3$ if the vertices of the triangle were allowed to coincide, i.e. $r_i = r_j$ were allowed. The second term $-3q$ means that when $r_i = r_j \neq r_k$, $r_j = r_k \neq r_i$ or $r_i = r_k \neq r_j$ (each has an occurrence factor q), the total occurrence factor, $3q$, must be subtracted from the first term. The last term, -1, means that when $r_i = r_j = r_k$, the occurrence factor 1 must also be excluded. The result may be expressed as

$$((\triangle)) = [[\triangle]] - 3((\,/\,)) - ((\cdot)) \tag{6}$$

where $((g))$, to be referred to as the *occurrence factor* of the graph g, is the high-temperature lattice constant of the graph g multiplied by its symmetry number, and $[[g]]$ indicates the corresponding quantity when some of the vertices of the graph may be coincided. The quantities $[[g]]$ will be called the *coincidable occurrence factor* of the graph g. Similarly we can obtain

$$\begin{aligned} ((\square)) &= [[\square]] - 4((\triangle)) - 2((\wedge)) - 7((\,/\,)) - ((\cdot)) \\ ((\langle|\rangle)) &= [[\langle|\rangle]] - 5((\triangle)) - ((\wedge)) - 7((\,/\,)) - ((\cdot)) \end{aligned} \tag{7}$$

etc.

The notations we shall adopt, and use hereafter, are the following: $g_{nx}^{(v)}$ refers to a graph of n lines and v vertices. (Either v or nx may be omitted for convenience). p_{nx}, the notation of Domb (1960), refers to the high-temperature lattice constant of the graph $g_{nx}^{(v)}$. The coincidable occurrence factor of the graph g_{nx} will be denoted as C_{nx}, i.e. $[[g_{nx}]] = C_{nx}$ and $((g_{nx})) = p_{nx} \times$ (symmetry number of g_{nx}).

Equations (6) and (7) can be stated generally by the following theorem.

Theorem 1. If $g_{nx}^{(v)}$ is a graph of v vertices

$$[[g_{nx}^{(v)}]] = \sum_{\mu,\, my} ((g_{nx}^{(v)}; g_{my}^{(\mu)})) \cdot ((g_{my}^{(\mu)})), \tag{8}$$

where the summation is taken over all graphs $g_{my}^{(\mu)}$ with number of vertices $\mu \leqslant v$, and number of lines $m \leqslant n$.

The matrix elements $((g_{nx}^{(v)}; g_{my}^{(\mu)}))$ are the numbers of ways of obtaining $g_{my}^{(\mu)}$ by bringing some vertices of the graph $g_{nx}^{(v)}$ into coincidence.

The quantities $((g_{nx}; g_{my})$ can be obtained by observation. As illustrated in figure 2, the graph g_{5a} reduces to g_3 when the vertices labelled 1 and 2 are brought into coincidence, or when the vertices 1 and 3 are merged, etc. We have $((g_{5a}; g_3)) = 5$. Similarly we obtain $((g_{8f}; g_4)) = 4$. Each graph shown on the right-hand side of figure 2 will be called a reduced graph of $g_{nx}^{(v)}$.

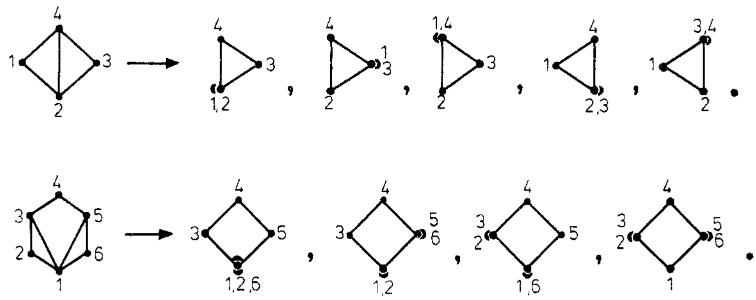


Figure 2. Determination of $((g_i; g_j))$ by observation. The upper graphs show that there are five methods of obtaining g_3 (a triangle) by bringing vertices of the left-hand graph, g_{5a}, into coincidence, i.e. $((g_{5a}; g_3)) = 5$. Similarly we obtain $((g_{8f}; g_4)) = 4$ from the lower graphs.

4.2. *Calculation of the coincidable occurrence factors C_{nx}*

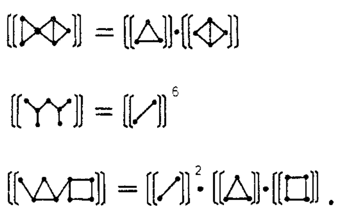
Consider a graph of $(v_1 + v_2 - 1)$ vertices. If we can break the graph into two separated graphs of v_1 and v_2 vertices by cutting the graph at a vertex, the graph is called an articulated graph (see p 5 of Domb and Green 1974) and the vertex is called a cutting point. We first prove the following theorem for articulated graphs.

Theorem 2. For an articulated graph $g_x^{(v_1+v_2-1)}$, if the graph breaks into two graphs $g_y^{(v_1)}$ and $g_z^{(v_2)}$ when cut at the cutting point, then

$$[[g_x^{(v_1+v_2-1)}]] = [[g_y^{(v_1)}]] \cdot [[g_z^{(v_2)}]]. \tag{9}$$

Proof. Choose the cutting point as the origin. As the vertices are allowed to coincide, the number of ways of locating the vertices of g_y on the lattice is independent of how the vertices of g_z are located on the lattice. The theorem then follows.

Theorem 2 simplifies our calculation of the coincidable occurrence factors. Only the multiply connected graphs (non-articulated graphs) need to be considered. There are many more articulated graphs than multiply connected graphs. The coincidable occurrence factors for articulated graphs are simply products of coincidable occurrence factors of multiply connected graphs. For example



(10)

Theorems 1 and 2 are valid for general lattice models. They apply to the DD model and regular lattices.

It is important to note that theorems 1 and 2 constitute a general means of expressing the high-temperature lattice constants of articulated graphs in terms of those of multiply connected graphs.

For the present model we have additional theorems.

Theorem 3. For a d-dimensional hypercubical lattice with the range of the interaction R defined by equation (1). The coincidable occurrence factor $C_{nx}(d, R)$ satisfies

$$C_{nx}(d, R) = [C_{nx}(1, R)]^d. \tag{11}$$

Proof. This theorem follows from an important feature of the hypercubical symmetric model that in summing over the lattice sites each lattice dimension can be performed independently.

By theorem 3 our calculation of C_{nx} on a general d-dimensional lattice is reduced to that of the one-dimensional lattice. For a graph of v vertices, $g_{nx}^{(v)}$, the coincidable occurrence factor on the one-dimensional lattice has the general expression

$$C_{nx}(1, R) = \sum_{r_2}\sum_{r_3}\dots\sum_{r_d}\{\prod_{\langle ij\rangle} f_{ij}\}, \tag{12}$$

where the summations go through all integers (the first vertex is fixed at the origin), the product is over all pairs of vertices and

$$f_{ij} = 0 \text{ if } |r_i - r_j| > R \text{ and the pairs of vertices } i \text{ and } j \text{ in the graph } g_{nx} \text{ are joined by a line,}$$
$$= 1 \text{ otherwise.}$$

Direct calculations of $C_{nx}(1, R)$ by performing the summations are very tedious, especially for graphs with more than five vertices. In our calculations, we find that for a connected graph of v vertices $C_{nx}(1, R)$ is a polynomial of degree $(v - 1)$ in R. This follows from the fact that the calculation involves a $(v - 1)$-fold summation (from $-R$ to R) of a step function which depends on the topology of the graph. The fact that $C_{nx}(1, R)$ is a polynomial in R provides a simple method to determine the function $C_{nx}(1, R)$ by machine computation.

For a fixed integral value of R the quantity $C_{nx}(1, R)$ can be determined by computer counting (Martin 1974). We evaluate $C_{nx}(1, 1)$, $C_{nx}(1, 2)$, ... $C_{nx}(1, v - 2)$ for a graph with v vertices. From these values and from the fact that $C_{nx}(1, 0) = 1$, $C_{nx}(1, R)$ is uniquely determined by determining the coefficients of the polynomial. To check the correctness of the polynomials we then evaluate the values $C_{nx}(1, v - 1)$, $C_{nx}(1, v)$, ... from the polynomials $C_{nx}(1, R)$ and compare them with those obtained from a direct computer counting. We get exact agreement for all graphs considered.

It is interesting to note that if we express $C_{nx}(1, R)$ as functions of $(2R + 1)$, $C_{nx}(1, R)$ is an even-power polynomial in $(2R + 1)$ for a graph with an odd number of vertices, and is an odd-power polynomial in $(2R + 1)$ for a graph with an even number of vertices.

The coincidable occurrence factors $C_{nx}(1, R)$ for multiply connected graphs with $n \leqslant 8$ are shown in Appendix 1.

4.3. High temperature lattice constants

As illustrated in figure 2 the matrix elements $((g_i^{(v)}; g_j^{(\mu)}))$ can be determined by observation. For a graph of v vertices the sum of the matrix elements $\sum\sum_{\mu j} ((g_i^{(v)}; g_j^{(\mu)}))$, which is the total number of reduced graphs of $g_i^{(v)}$, increases very rapidly with v. For instance the total numbers of reduced graphs are 2, 5, 15, 52, 203, 877 and 4140 respectively for graphs with $v = 2, 3, 4, 5, 6, 7$ and 8. We have determined those elements for graphs up to eight lines.

Equation (8) can easily be solved to express $((g_{nx}))$ in terms of $[[g_{my}]]$. We obtain

$$
\begin{aligned}
((\diagup)) &= [[\diagup]] - [[\cdot]], \\
((\wedge)) &= [[\wedge]] - 3[[\diagup]] + 2[[\cdot]], \\
((\triangle)) &= [[\triangle]] - 3[[\diagup]] + 2[[\cdot]], \\
((\mathrm{N})) &= [[\mathrm{N}]] - [[\triangle]] - 5[[\wedge]] + 11[[\diagup]] - 6[[\cdot]], \\
((\square)) &= [[\square]] - 4[[\triangle]] - 2[[\wedge]] + 11[[\diagup]] - 6[[\cdot]], \\
&\vdots
\end{aligned}
\tag{13}
$$

If we denote $s = s(d, R) = (2R + 1)^d$, the high-temperature lattice constants for the multiply connected graphs are:

$$
\begin{aligned}
q &= s - 1, \\
6p_3 &= C_3 - 3s + 2 \\
8p_4 &= C_4 - 4C_3 - 2s^2 + 11s - 6 \\
10p_5 &= C_5 - 5C_4 - C_3(5s - 20) + 15s^2 - 50s + 24 \\
4p_{5a} &= C_{5a} - 5C_3 - s^2 + 11s - 6 \\
12p_6 &= C_6 - 3C_{6c} - 6C_5 + 9C_{5a} - C_4(6s - 27) \\
&\quad + C_3(42s - 120) + 7s^3 - 105s^2 + 274s - 120 \\
&\vdots
\end{aligned}
\tag{14}
$$

The expressions of p_{nx} become very lengthy for higher-order graphs. Rather than list these expressions, we show in Appendix 2 the high temperature lattice constants p_{nx} for $n \leqslant 8$ for some values of d and R.

5. Anisotropy in the range of the interaction

In the above discussion the range of the interaction is isotropic in each dimension. It is straightforward to extend the model to an anisotropic case. As mentioned in § 4, theorems 1 and 2 are generally valid. They hold true for the anisotropic case. To derive the high-temperature lattice constants for the anisotropic case, only the coincidable occurrence factors need to be determined. We have the following theorem.

Theorem 4. In a $(d_1 + d_2 + d_3 + \ldots)$-dimensional hypercubical lattice if the range of the interaction is R_1 in d_1 lattice directions, and is R_2 in d_2 lattice directions, etc., the coincidable occurrence factor is

$$
C_{nx} = [C_{nx}(1, R_1)]^{d_1} \cdot [C_{nx}(1, R_2)]^{d_2} \ldots . \tag{15}
$$

The proof of the theorem is the same as that of theorem 3.

The high-temperature lattice constants are then calculated from equation (14). For instance, the coordination number q becomes $-1 + (2R_1 + 1)^{d_1} . (2R_2 + 1)^{d_2} \ldots$, and equation (5*a*) for p_3 becomes

$$6p_3 = (3R_1^2 + 3R_1 + 1)^{d_1} . (3R_2^2 + 3R_2 + 1)^{d_2} \ldots - 3(2R_1 + 1)^{d_1} . (2R_2 + 1)^{d_2} \ldots + 2. \qquad (16)$$

6. Extension to non-integral values of *d* and *R*

In §4 the lattice constants are calculated for integral values of d and R. The high-temperature lattice constants are linear combinations of some polynomials in $(2R + 1)$ raised to the power of d. These polynomials $C_{nx}(1, R)$ are non-negative as long as $R \geqslant 0$. This enables us to extend d and R from integers to real numbers.

The physical meaning for non-integral values of R may be interpreted as that only a certain fraction of the outer shell spins are interacting with the spin at the centre of a hypercube. For instance, referring to figure 3 which shows a two-dimensional lattice with $R = 2{\cdot}5$, all spins in the two inner shells (marked $\bigcirc$) interact with the spin at the centre, while only one quarter ($0{\cdot}5 \times 0{\cdot}5$) of the spins at the corner of the third shell (marked $\oplus$), and one half ($0{\cdot}5$) of the other spins in the third shell (marked $\oslash$) interact with the centre spin.

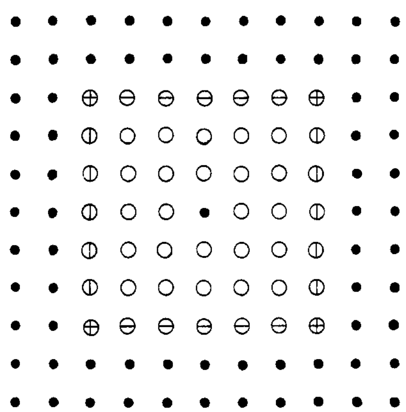


Figure 3. In a 2-dimensional lattice with $R = 2{\cdot}5$ all spins in the two inner shells (marked $\bigcirc$) interact with the spin at the centre. One quarter of the spins at the corners of the third shell (marked $\oplus$), and one half of the other spins in the third shell (marked $\oslash$) interact with the centre spin.

It is important to point out that the lattice constants p_{nx} must be non-negative and non-decreasing functions of d and R. The functions p_{nx} in equations (14), however, are oscillating and may be negative when the coordination number q (or the range of the interaction R) is small. In this region the function p_{nx} given by equations (14) must be modified in a reasonable way. As an example, figure 4 shows p_{5a} as functions of $q = (2R + 1)^d - 1$, for $d = 2$, and 3. For $d = 2$, the function p_{5a} has a maximum real root at $R = 0{\cdot}64$ (or $q = 4{\cdot}22$), and has an oscillating tail (shown by dashed line in figure 4) for $q < 4{\cdot}22$. Consequently the function $p_{5a}(d = 2, R)$ must be replaced by zero

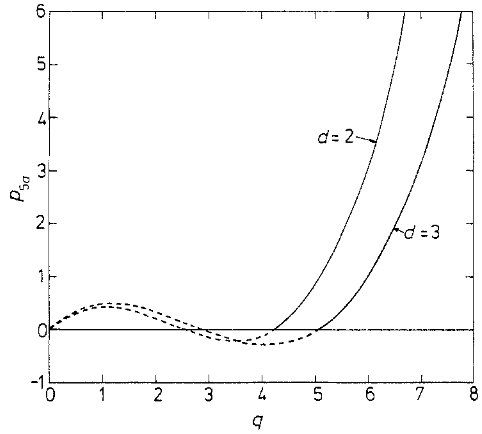


Figure 4. The function $p_{5_a}(d, R)$, equation (14); for $d = 2$, and 3. The oscillating tails, indicated by broken lines, must be cut off (replaced by zero).

for $R < 0{\cdot}64$. Similarly when $d = 3$, the function $p_{5_a}(R)$ must be replaced by zero for $R < 0{\cdot}41$ ($q < 6{\cdot}06$). The region for which the function p_{5_a} given by equation (14) must be replaced by zero is shown in figure 5. The region that all the functions $p_{nx}(d, R)$ for closed graphs (any vertex of the graph has at least two lines connected to it) with $n \leqslant 8$ are positive and increasing is also shown in figure 5.

Although the magnitudes of the oscillating tails are very small as compared with the lattice constants of Cayley trees and polygons, the cut-off process for the oscillating tail is important in the high-temperature series expansion when the coordination number is very small. The cut-off process is more important in the low-temperature series expansions when the low-temperature lattice constants are expressed in terms of the high-temperature lattice constants. We will discuss this point further when we consider the low-temperature series expansion for the present lattice in a later publication.

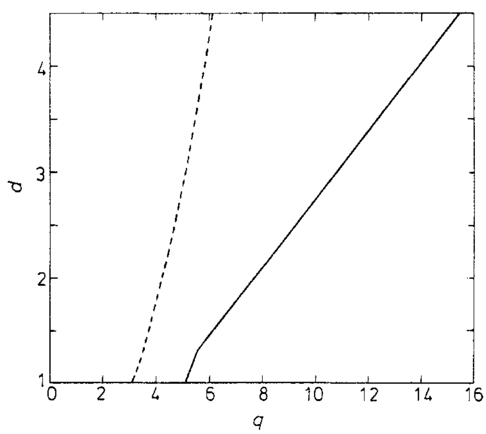


Figure 5. The outermost locus of the real roots of $p_{5_a}(d, R) = 0$ is indicated by the broken line. For a point (d, R) on its left side, $p_{5_a}(d, R)$, equation (14), should be replaced by zero. On the right side of the solid line $p_{nx}(d, R)$ for closed graphs with $n \leqslant 8$ are all positive and increasing functions of d and R.

For a well behaved lattice, there are additional restrictions to the occurrence factors $((g))$ (high temperature lattice constant × symmetry number). Consider two graphs of v vertices $g_{nx}^{(v)}$ and $g_{n+1,y}^{(v)}$, where $g_{n+1,y}$ is obtained by adding one line to g_{nx}. Because the positions of the vertices in $g_{n+1,y}$ are more restricted than those in g_{nx}, it is necessary that

$$((g_{n+1,y}^{(v)})) \leqslant ((g_{n,x}^{(v)})) \tag{17}$$

for example, $((g_{5a})) \leqslant ((g_4))$ and $((g_{8q})) \leqslant ((g_{7h}))$. In the present model, when d and R are extended to non-integral values, inequalities (17) are generally satisfied except for a few cases. Figure 6 shows the functional dependence on q at $d = 3$ for $((g_{8q}))$ and $((g_{7h}))$. We see that in the region $8{\cdot}3 < q < 9{\cdot}4$, or $0{\cdot}55 < R < 0{\cdot}59$ the inequality $((g_{8q})) \leqslant ((g_{7h}))$ is not satisfied.

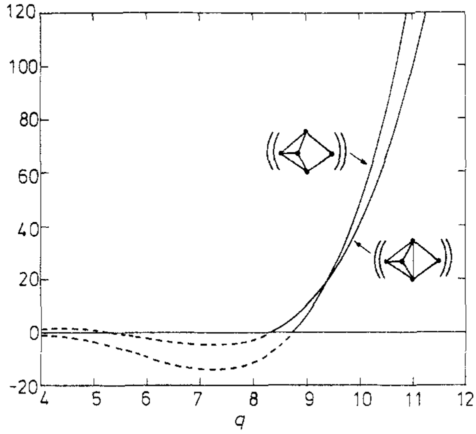


Figure 6. The occurrence factors $((g_{7h})) = 4p_{7h}$ and $((g_{8q})) = 4p_{8q}$ as functions of q for $d = 3$. The inequality $((g_{8q})) \leqslant ((g_{7h}))$ is violated when $8{\cdot}3 < q < 9{\cdot}4$.

It may be desirable to modify the functions $p_{nx}(d, R)$ such that inequalities (17) are satisfied in all cases. Since in each case when any of the inequalities (17) is violated the difference between $((g_{n+1,y}))$ and $((g_{nx}))$ is very small as compared with other occurrence factors, the modifications of p_{nx} are unnecessary when the series expansions are to be used for the investigation of critical properties.

7. Conclusions

We have introduced a generalised equivalent-neighbour model in which the range of the interaction is hypercubically symmetric (instead of spherically symmetric). The range of the interaction R and the lattice dimensionality d are extended from positive integers to real values. The present lattice can be extended to an anisotropic one in which the ranges of interactions are different in different lattice directions.

We introduced the concept of the coincidable occurrence factors. The conventional high-temperature lattice constants p_{nx} as functions of d and R are derived through the calculations of the coincidable occurrence factors C_{nx}. General expressions of C_{nx} and some numerical values of p_{nx} are given for $n \leqslant 8$. Only the high-temperature lattice constants are derived in this paper. The low-temperature lattice constants of the present model can also be obtained since they are linearly related to the high-temperature lattice constants (Sykes *et al* 1966).

The present model can be applied to investigate the dependence of critical properties on lattice dimensionality d, on the coordination number q (or the range of interaction R), as well as on the lattice anisotropy. An application of the present lattice to the Ising model has been reported briefly by the authors (Chen and Lee 1979). Fuller details of applications of the present model will be given in subsequent papers.

Appendix 1. Coincidable occurrence factors

The coincidable occurrence factors for multiply connected (non-articulated) graphs on the one-dimensional lattice, $C_{nx}(1, R)$, are given below. Those on a general d-dimensional lattice are related to $C_{nx}(1, R)$ by equations (11) and (15). The coincidable occurrence factors for articulated graph are related to those for multiply connected graphs by equation (9). Here

$$Y = 2R + 1,$$
$$C_3(1, R) = (3Y^2 + 1)/4$$
$$C_4(1, R) = (2Y^3 + Y)/3$$
$$C_5(1, R) = (115Y^4 + 50Y^2 + 27)/192$$
$$C_{5a}(1, R) = (7Y^3 + 5Y)/12$$
$$C_6(1, R) = (11Y^5 + 5Y^3 + 4Y)/20$$
$$C_{6a}(1, R) = (Y^4 + Y^2)/2$$
$$C_{6b}(1, R) = (49Y^4 + 38Y^2 + 9)/96$$
$$C_{6d}(1, R) = (Y^3 + Y)/2$$
$$C_7(1, R) = (5887Y^6 + 2695Y^4 + 1813Y^2 + 1125)/11520$$
$$C_{7a}(1, R) = (27Y^5 + 25Y^3 + 8Y)/60$$
$$C_{7b}(1, R) = (439Y^5 + 330Y^3 + 191Y)/960$$
$$C_{7c}(1, R) = (29Y^4 + 30Y^2 + 5)/64$$
$$C_{7f}(1, R) = (419Y^5 + 370Y^3 + 171Y)/960$$
$$C_{7g}(1, R) = (15Y^4 + 18Y^2 - 1)/32$$
$$C_{7h}(1, R) = (41Y^4 + 46Y^2 + 9)/96$$
$$C_8(1, R) = (151Y^7 + 70Y^5 + 49Y^3 + 45Y)/315$$
$$C_{8a}(1, R) = (1607Y^6 + 1235Y^4 + 773Y^2 + 225)/3840$$
$$C_{8b}(1, R) = (4649Y^6 + 4145Y^4 + 2051Y^2 + 675)/11520$$
$$C_{8c}(1, R) = (71Y^6 + 65Y^4 + 44Y^2)/180$$
$$C_{8d}(1, R) = (1477Y^6 + 1245Y^4 + 983Y^2 + 135)/3840$$
$$C_{8e}(1, R) = (381Y^5 + 430Y^3 + 149Y)/960$$
$$C_{8f}(1, R) = (47Y^5 + 50Y^3 + 23Y)/120$$
$$C_{8p}(1, R) = (37Y^5 + 46Y^3 + 13Y)/96$$
$$C_{8q}(1, R) = (19Y^4 + 26Y^2 + 3)/48$$
$$C_{8r}(1, R) = (6Y^5 + 10Y^3 - Y)/15$$

$C_{8s}(1, R) = (97Y^5 + 110Y^3 + 33Y)/240$
$C_{8t}(1, R) = (11Y^5 + 15Y^3 + 4Y)/30.$

Appendix 2. High-temperature lattice constants p_{nx} for some values of d and R

Following each set of the values of d and R the entries in the first line are the lattice constants q, p_3, p_4, p_5, p_{5a}, p_6, p_{6a}, p_{6b}, p_{6c} and p_{6d}; the second line p_7, p_{7a}, p_{7b}, p_{7c}, p_{7d}, p_{7e}, p_{7f}, p_{7g} and p_{7h}; the third line p_8, p_{8a}, p_{8b}, p_{8c}, p_{8d}, p_{8e}, p_{8f}, p_{8g} and p_{8h}; and the last line p_{8j}, p_{8k}, p_{8l}, p_{8m}, p_{8p}, p_{8q}, p_{8r}, p_{8s} and p_{8t}.

$d = 1, R = 2$:
4, 1, 1, 1, 1, 1, 0, 2, 1, 0;
1, 1, 2, 1, 4, 2, 0, 0, 0;
1, 2, 2, 0, 0, 2, 0, 2, 1;
8, 9, 0, 2, 0, 0, 0, 0, 0.

$d = 1, R = 3$:
6, 3, 7, 15, 10, 31, 6, 46, 16, 1;
65, 45, 100, 31, 73, 70, 31, 5, 11;
137, 224, 186, 60, 43, 118, 48, 148, 75;
352, 295, 36, 58, 36, 8, 1, 24, 7.

$d = 1{\cdot}5, R = 1{\cdot}5$:
7, 3·479, 9·295, 24·644, 11·795, 69·815, 7·842, 65·845, 25·662, 0·958;
210·107, 88·882, 194·325, 39·127, 152·562, 147·932, 56·271, 6·282, 11·718;
667·820, 628·087, 503·205, 161·801, 122·473, 207·146, 82·853, 418·941, 204·760;
901·817, 787·033, 71·595, 105·805, 57·541, 7·785, 1·475, 40·904, 10·919.

$d = 2, R = 1$:
8, 4, 12, 36, 14, 126, 10, 88, 38, 1;
476, 150, 312, 48, 274, 268, 84, 8, 12;
1941, 1280, 1016, 316, 230, 320, 120, 888, 450;
1896, 1712, 124, 172, 80, 8, 2, 60, 16.

$d = 2, R = 2$:
24, 48, 600, 7928, 866, 112686, 4074, 26452, 8564, 97;
1672020, 199126, 417060, 19448, 189206, 266608, 177980, 3482, 8162;
25677769, 6856592, 6127528, 2789090, 2563118, 571424, 272332, 4151396, 2024474;
5874804, 4096878, 184992, 193052, 254504, 6700, 11238, 146236, 55308.

$d = 2{\cdot}5, R = 1{\cdot}0$:
14·558, 14·146, 91·809, 608·488, 116·460, 4559·269, 246·245, 1722·667, 646·361, 10·793;
36415·179; 6467·877, 13477·582, 1099·131, 8646·387, 10233·281, 4978·712, 201·231, 396·851;

308471·530, 117125·135, 98763·058, 40067·784, 33560·326, 15706·609, 6956·282, 78414·759, 39482·780;

137267·967, 110132·333, 5883·729, 6588·693, 6016·309, 302·911, 257·501, 3820·089, 1214·867.

$d = 3, R = 0{\cdot}5$:

7, 2·055, 4·086, 6·516, 3·184, 24·222, 0, 8·988, 9·549, 0·043;

72·401, 29·483, 35·788, 1·733, 67·534, 63·098, 0, 0·094, 0;

349·412, 93·812, 70·735, 0, 0, 55·529, 7·786, 94·250, 66·307;

277·682, 346·805, 21·966, 30·825, 2·964, 0, 0, 0·707, 5·285.

$d = 3, R = 1$:

26, 44, 540, 6852, 690, 99078, 3026, 19992, 7266, 67;

1524156, 148602, 306600, 13200, 179574, 226116, 122616, 2480, 5088;

24833163, 5162904, 4481592, 1943964, 1686978, 374928, 171456, 3425880, 1730946;

5592768, 4331100, 137316, 144804, 157608, 3984, 7350, 95916, 32664.

$d = 4, R = 1$:

80, 360, 13560, 528360, 16460, 24103020, 233572, 1482352, 549740, 1546;

1175749976, 35119068, 71600880, 957408, 43390516, 54389176, 29622984, 187760, 373848;

61178526258, 3832525184, 3368786288, 1506889624, 1311723548, 86528768, 39814320, 2610886032, 1338243876;

4294121904, 3395713472, 32427832, 32988568, 38028128, 290096, 1936380, 22934232, 7894384.

$d = 5, R = 0{\cdot}5$:

31, 44·765, 577·955, 6875·059, 635·472, 106208·218, 2674·760, 17646·549, 7679·536, 53·977;

1727613·801, 139188·953, 263486·018, 10432·699, 234124·649, 258508·489, 95431·118, 2142·529, 3504·177;

31482380·192, 4693971·924, 4032651·024, 1613925·002, 1220286·080, 298201·806, 122691·902, 3708869·292, 2154800·118;

7837969·554, 6910213·105, 133771·868, 141659·975, 111911·591, 2795·142, 5619·470, 74091·324, 22675·622.

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