RELATIVISTIC SELF-SIMILAR DISKS

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ABSTRACT

We formulate and solve by semianalytic means the axisymmetric equilibria of relativistic self-similar disks of infinitesimal vertical thickness. These disks are supported in the horizontal directions against their self-gravity by a combination of isothermal (two-dimensional) pressure and a flat rotation curve. The dragging of inertial frames restricts possible solutions to rotation speeds that are always less than 0.438 times the speed of light, a result first obtained by Lynden-Bell & Pineault in 1978 for a cold disk. We show that prograde circular orbits of massive test particles exist and are stable for all of our model disks but that retrograde circular orbits cannot be maintained with particle velocities less than the speed of light once the disk develops an ergoregion. We also compute photon trajectories, planar and non-planar, in the resulting spacetime for disks with and without ergoregions. We find that all photon orbits, except for a set of measure zero, tend to be focused by the gravity of the flattened mass-energy distribution toward the plane of the disk. This result suggests that strongly relativistic, rapidly rotating, compact objects may have difficulty ejecting collimated beams of matter or light along the rotation axes until the flows get well beyond the flattened parts of the relativistic mass distribution (which cannot happen in the self-similar models considered in this paper).

Subject headings: accretion, accretion disks — black hole physics — relativity

1. INTRODUCTION

It has been said that there are basically only two kinds of self-gravitating objects in astronomy, spheres and disks. It has also been said that general relativity is so beautiful, it has to be right. Thus, spheres and disks should be of as much natural interest to relativists as to Newtonian dynamicists. Yet, because the beauty of general relativity comes at the steep price of great mathematical difficulty, for many decades, the only known solutions to Einstein’s field equations were ones possessing spherical symmetry. Any analytical attempts to study realistic rotating bodies relied on perturbation theory and the assumption of low angular momentum. It took astrophysicists nearly 50 years after Einstein first wrote down the full form of his theory to find an asymptotically flat vacuum solution that has nontrivial angular momentum (Kerr 1963) and another 33 years to construct appropriate interior solutions (see, e.g., Pichon & Lynden-Bell 1996). Some progress has been made in the study of relativistically rotating, axisymmetric objects with finite physical extension but mostly in the limit when the disk is cold (Bardeen & Wagoner 1971; Lynden-Bell & Pineault 1978b) or when the material in it is taken in the form of two equal, collisionless, counterrotating sheets (Lynden-Bell & Pineault 1978a; Lemou 1989). The first case results in the mathematical simplification that the number of unknown metric functions reduces to three, the second, in the elimination of the dragging of inertial frames. However, the Newtonian analogs to cold disks are fraught with fierce dynamical instabilities (see, e.g., Binney & Tremaine 1987). Such models cannot represent good approximations for realistic astrophysical systems (e.g., spiral galaxies [see Bertin & Lin 1996] or protoplanetary disks [see Adams & Lin 1993]). Counterrotating disks can avoid instability if they are sufficiently hot, but since such configurations have to arise as stellar-dynamical (collisionless) rather than gas-dynamical (collisional) systems, relativistic analogs may have difficulty reaching the requisite degree of physical compactness.

In the Newtonian studies of self-gravitating disks, devices that have proven to have great mathematical utility are the assumptions of complete flattening and self-similarity (see, e.g., Mestel 1963; Zang 1976; Toomre 1977; Shu et al. 2000). Razor-thin disks whose surface densities are power laws in radius r but which need not possess axial symmetry have solutions that can be found by analytical or semianalytical means (i.e., involving nothing worse than the numerical integration of ordinary differential equations [ODEs]; see, e.g., Syer & Tremaine 1996; Galli et al. 2001). The gravitational collapse of such Newtonian models has elegant self-similar properties in spacetime (see, e.g., the study of Li & Shu 1996 of the collapse of the axisymmetric singular isothermal disk). The relativistic analogs of such gravitational collapses, axisymmetric and nonaxisymmetric, could lend valuable insight into issues of great contemporary interest in general relativity, such as the efficiency of gravitational radiation or the possible formation of naked singularities.

As preparatory work toward such applications, we wish to extend the work of Lynden-Bell & Pineault (1978b) on cold, axisymmetric, relativistically rotating disks to obtain semianalytical solutions of a family of relativistic disks parameterized by constant isothermal sound and rotation speeds, a and V, respectively. Self-similarity is then dimensionally still possible in the relativistic regime because relativity introduces only one other constant, the speed of light c, with the same units of velocity. Since no power of G, the universal gravitational constant, combined with c (or a or V) can yield a quantity with the dimensions of length (or time), it becomes natural and feasible to look for unbounded disk solutions in which the surface density
varies as a power of some appropriately chosen radial coordinate $r$. The mathematical consequences of this basic idea are developed in the paper as follows. Sections 2 and 3 derive the equations of axisymmetric stationary spacetime generated by a rotating disk. Section 4 describes our numerical strategy for the solution of the resulting ODEs. Section 5 gives the results of the numerical integrations, and it also explores some properties of massive test particles placed in circular orbits in the disk plane. Section 6 considers the orbits of massless test particles (photons or neutrinos) in the general spacetimes of our models. Finally, in §7 we offer our conclusions and speculations.

2. DIMENSIONLESS BASIC EQUATIONS

2.1. Elementary Dimensional Considerations

We characterize isothermal disks with flat rotation curves by two dimensionless parameters: the linear rotation velocity as a fraction of the speed of light, $v \equiv V/c$, and the square of the isothermal sound speed as a fraction of $c^2$, $\gamma \equiv (a/c)^2$. We nondimensionalize by adopting the unit of mass per unit length as $c^2/G = 1.35 \times 10^{28}$ g cm$^{-1}$ = 0.677 $M_\odot$ km s$^{-1}$. Notice that with $c$ and $G$ alone, we cannot define a characteristic mass per unit area (surface density) nor can we define a characteristic length. Thus, if $r$ is a coordinate radius, with the physical units of length, we are naturally interested in disks with surface densities that are proportional to $c^2/Gr$, i.e., with surface densities that are inversely proportional to one power of $r$.

In general relativity it is possible to accomplish the equivalent nondimensionalization by working in the geometrical system of units where $c = G = 1$. It is also possible, of course, to choose a radial coordinate $r_{LP}$ that does not have the units of length (see, e.g., Lynden-Bell & Pineault 1978b; Lemos 1989). However, to maintain self-similarity in the problem, $r_{LP}$ can be, at best, only some power of $1/k$; see below) of $r$. Using $r_{LP}$ allows us to specify in advance that in going from pole to pole along a locus $r_{LP} = \text{constant}$ (for a slice at constant $\phi$ and $t$), the associated polar angle $\theta$ ranges from 0 to $\pi$, with the disk midplane located by symmetry at $\theta = \pi/2$. However, this adherence to normal convention comes only at the expense of making $k$ a nonlinear eigenvalue of the problem. Using $r$ and absorbing $k$ into the definition of $\theta$ eliminates the need for a complicated numerical procedure to find the value of $k$, but it puts the location of the midplane at a polar angle $\theta = \theta_0 \neq \pi/2$. The former represents a considerable computational advantage, whereas the latter serves as a useful reminder that if we choose a coordinate $r$ such that $\mathcal{M} \propto r$ represents the physical radial distance from the origin to the point $(r, \theta, \phi)$ in the disk midplane (with $\mathcal{M}$ having operational meaning as a proper radius because the origin is only mildly singular), then the distance along constant $\mathcal{M}$ (again for a slice at constant $\phi$ and $t$) from the midplane to the pole may not equal $\pi \mathcal{M}/2$ because of the distortion of the spatial geometry produced by the flattened mass distribution.

2.2. The Metric

Without losing generality, the metric that is stationary, axisymmetric, and invariant under $\phi \to -\phi$ and $t \to -t$ may be written in geometrical units as

$$ds^2 = -e^{2\nu} dt^2 + B^2 e^{-2\nu} r^2 \sin^2 \theta (d\phi - \omega dt)^2 + e^{2(\nu - \varphi)} (dr^2 + r^2 d\theta^2),$$

where $v, B, \nu$, and $\mu$ are, in general, functions of $r$ and $\theta$.

Since there is no fundamental unit of length, one might naively conclude that $v, B, \nu$, and $\mu$ are functions of $\theta$ only in a self-similar disk. This conclusion is premature and false.

In the weak field limit, when the surface density and the rotation velocity are small, a cold Mestel disk of infinite extent has the associated gravitational potential

$$-\nu = \Phi = -v^2 \ln \left( \frac{r}{D} \left( 1 + |\cos \theta| \right) \right),$$

where $D$ is a fiducial length scale that contributes only an added constant to the potential and thus enters nowhere else in the problem. We will discard $D$ in what follows. In the Newtonian limit, therefore, $e^{\nu} \propto n^2$, where $n \approx v^2$ when $v^2 \ll 1$ (with $n$ having a different dependence on $v$ and possibly also on $\gamma$ when the rotation and isothermal sound speeds in the disk are not small compared to $c$). This behavior—the nonpredetermined power law of the gravitational distortion of time and the spatial geometry by the disk’s self-gravity—is the source of the scaling relationships described by Lynden-Bell & Pineault (1978b) and Lemos (1989). According to the nomenclature of Barenblatt (1976), the situation is an example of self-similarity of the second kind.

Equation (1) allows a spacetime that is self-similar under the transformation $r \to \alpha r, t \to \alpha^{-1} t$. Then $ds^2 \propto \alpha^2 ds^2$, and we can write the metric for $0 < \theta < \theta_0$ as

$$ds^2 = -r^2 e^{2\nu} dt^2 + r^2 e^{2\nu - N} (d\phi - \omega^{-1} e^{-N} Q dt)^2 + e^{2(\nu - N)} (dr^2 + r^2 d\theta^2),$$

where $N, P, Q,$ and $Z$ are functions of $\theta$ only. In the above metric, $n$ is a pure number between 0 and 1 that measures the depth of the disk’s gravitational well. In particular, photons that emerge from the origin and reach some finite $r$ have frequencies that are infinitely redshifted from their starting values. Self-similarity then implies that the same infinite redshift applies to photons that originate at any finite $r$ and try to propagate to infinity (see §5). In a certain sense, therefore, the system constitutes an incipient black hole, one that will presumably acquire a growing point mass at the origin, with an accompanying horizon, if the disk is unstable and undergoes inside-out gravitational collapse (see Li & Shu 1996 for a Newtonian analog).

It is instructive to compare our metric to the one used by Lynden-Bell & Pineault (1978b) and Lemos (1989):

$$ds^2 = -r^2_{LP} e^{N} dt^2 + r^2_{LP} e^{2P - N} \times (d\phi - \omega^{-1} e^{-N} Q dt)^2 + (2e^{P - 1}) e^{2N - N} (dr^2_{LP} + r^2_{LP} d\theta^2).$$

The relationship between the two coordinate conventions can be made explicit by the following transformation:

$$r = r^2_{LP}, \quad \theta = k \chi, \quad n = \frac{m}{k}, \quad Z = Z_{LP} - \ln k.$$

While the two metrics are completely equivalent, our choice turns out to be more advantageous in numerical implementation. For Lemos and Lynden-Bell & Pineault, the disk is located at $\chi = \pi/2$ and $m$ and $k$ are eigenvalues of the problem. To find their values and the scaling of rotation velocity with density and pressure requires a three-dimensional shooting method, a nontrivial numerical task.
For our metric, we have only one eigenvalue, \( n \). The other degree of freedom is embedded in the location of the disk, which we can find by satisfying certain jump conditions when we cross from the top to bottom surface. Later on, we will reparameterize the solution space to avoid even having to find \( n \) as an eigenvalue.

Define the orthonormal tetrads for the locally nonrotating observer (Lemos 1989):
\[
e_{(0)}^\mu = (r^{-n}e^{-N/2}, r^{-1}Qe^{N/2-p}, 0, 0), \quad (6)
\]
\[
e_{(1)}^\mu = (0, r^{-1}e^{N/2-p}, 0, 0), \quad (7)
\]
\[
e_{(2)}^\mu = (0, 0, e^{(N-z)/2}, 0), \quad (8)
\]
\[
e_{(3)}^\mu = (0, 0, 0, r^{-1}e^{(N-z)/2}). \quad (9)
\]
The indices in parentheses label the basis vectors in \((t, \phi, r, \theta)\) and are raised and lowered with the flat Minkowski metric \( \eta = \text{diag}(-1, 1, 1, 1) \). The nontrivial Ricci components are
\[
2R_{00}r^2e^{N} = N_{\theta\theta} + N_\phi P_\phi + 2n(1+n) - Q^2 \times \{[\ln Q]_0 - P_\theta + N_\theta^2 + (1-n)^2\}, \quad (10)
\]
\[
2R_{01}r^2e^{N} = Q_{\theta\theta} + Q_\phi P_\phi - Q[P_{\phi\theta} - N_{\theta\theta} + (P_\theta - N_\theta)^2 \\
+ P_\phi(P_\phi - N_\phi) + 2(1-n)], \quad (11)
\]
\[
R_{00} - R_{11}r^2e^{N} = P_{\phi\phi} + P_\theta^2 + (n+1)^2, \quad (12)
\]
\[
2R_{22}r^2e^{N} = N_{\theta\theta} - Z_\theta + P_\phi(N_\theta - Z_\theta) \\
+ 2n(1-n) + Q^2(1-n^2), \quad (13)
\]
\[
2R_{23}r^2e^{N} = (n+1)Z_\theta - 2nN_\theta + Q^2(1-n) \\
\times [P_\theta - (\ln Q)_0 - N_\theta], \quad (14)
\]
\[
2[R_{33} + R_{00} - R_{11} - R_{22}]r^2e^{N} = 2P_\phi Z_\theta - N_\phi^2 \\
+ 4n^2 + Q^2\{(N_\theta + (\ln Q)_0 - P_\theta)^2 - (1-n)^2\}. \quad (15)
\]

### 2.3. Matter

For the stationary thin disk, there is no radial or vertical motion. Hence, we may write the four-velocity as
\[
u^\mu = r^{-n}e^{-N/2}(1 - v^2)^{-1/2}(1, \Omega, 0, 0), \quad (16)
\]
where
\[
\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} \quad (17)
\]
is the coordinate angular velocity and
\[
v = r^{-n}e^{-N/2}\Omega - Q \quad (18)
\]
is the linear velocity of the fluid in the locally nonrotating frame. The physical significance of this quantity becomes clear when we project the four-velocity onto the locally nonrotating frame defined by the tetrad
\[
u^{(a)} = \left(\frac{1}{\sqrt{1-v^2}}, \frac{v}{\sqrt{1-v^2}}, 0, 0\right); \quad (19)
\]
where a subscript "0" denotes the value on the disk at \( \theta = \theta_0 \). Define a further rescaling,
\[
\Theta \equiv (1 + n)\theta
\]
and let a prime denote differentiation with respect to \( \Theta \). Coupled to matter, the equation for \( P \) takes the form
\[
P'' + P^2 + 1 = \tilde{\rho}_r \delta(\Theta - \Theta_0) .
\]
Away from the disk, we can solve this equation subject to the boundary condition \( e^{\nu(0)} = 0 \) (so that a circle around the axis will have vanishingly small circumference as \( \Theta \to 0 \)):
\[
P = \ln(\sin \Theta) + C, \quad P' = \cot \Theta .
\]
The constant \( C \) remains arbitrary, which enables us to set the boundary condition for other metric functions later. The solution (eq. [31]) is only valid in the range \( 0 < \Theta < \Theta_0 \) where \( P \) is differentiable. For \( \Theta > \Theta_0 \), we can obtain the solution simply through symmetry considerations (i.e., the metric functions are even about the disk, while the derivatives are odd). Integrating equation (30) across \( \Theta = \Theta_0 \) and combining the result with the second relation of equation (31), we get
\[
-2 \cot \Theta_0 = \tilde{\rho}_r .
\]

Let us confine our attention to \( 0 \leq \Theta \leq \Theta_0 \). The rest of the field equations become
\[
N'' + N'P' + \frac{2n}{1 + n} - Q^2 \\
\times \left\{ \left[ \ln(Q)' - P' + N' \right]^2 + \left( \frac{1 - n}{1 + n} \right)^2 \right\}
= \left[ \tilde{\rho}_r + (\tilde{\epsilon} + \tilde{\rho}_\phi) \frac{1 + v^2}{1 - v^2} \right] \Delta ,
\]
and the constraint equation
\[
[QP' - Q' - QN']^2 - Q^4 \left( \frac{1 - n}{1 + n} \right)^2 + \frac{4n^2}{1 + n} P'N' \\
- 2QP' \frac{1 - n}{1 + n} [QP' - Q' - QN'] - N^2 \\
+ \frac{4n^2}{(1 + n)^2} = 0 .
\]
Now \( Z \) has been completely decoupled from \( Q \) and \( N \). We may use equation (39) to decouple \( Q \) from \( N \) as well, but this offers little advantage for numerical purposes.

4. NUMERICAL IMPLEMENTATION

4.1. Boundary Conditions

We have already discussed the boundary condition for \( P \). Unless \( r = 0 \), we expect space to be regular on the axis, which means the Ricci tensor remains finite there (and thus, Riemann normal coordinates exist there). On the pole, \( P' = \cot \Theta \) diverges as \( \Theta \to 1 \). In order to have a regular solution for equation (39) there, we require
\[
N' = 0, \quad Q = 0, \quad \text{at} \quad \Theta = 0 .
\]
The first requirement prevents the \( \Theta \) geometry from having a cusp at the pole; the second discounts frame dragging on the rotation axis. One might naively expect \( Q = 0 \) on the pole as well from the term \( QP' \) in equation (38); however, this singularity is cancelled by \( QP'' \). In fact, we can use \( Q' \) on the axis to parameterize the solution space. With proper rescaling of \( \tau, r, \) and \( v', \) we can set \( N = 0 \) on the pole (which means \( r' dt \) is the interval of proper time of an observer at \( \Theta = 0 \)).

On the other hand, the delta functions on the right-hand sides of the governing ODEs (37) and (38) signal a discontinuity in the derivatives of metric coefficients when we cross the plane of the disk. Similar to the method we used to find \( P \), we integrate across the disk and assume that the geometry is symmetric about the disk plane. Equations (37) and (38) then yield the following boundary conditions at \( \Theta = \Theta_0 \):
\[
N_0 = -\frac{1}{2} \left[ \tilde{\rho}_r + (\tilde{\epsilon} + \tilde{\rho}_\phi) \frac{1 + v^2}{1 - v^2} \right] ,
\]
\[
Q'_0 = \frac{1}{2} (\tilde{\epsilon} + \tilde{\rho}_\phi) Q_0 + Q_0 v^2 + 2v .
\]

4.2. Scalar Two-Dimensional Pressure and Method of Solution

For simplicity, we adopt a planar isotropic equation of state,
\[
p_r = p_\phi = \gamma e ,
\]
where \( \gamma \) is the isothermal sound speed squared, as usual. Then, on the disk, equations (27), (32), and (41) imply
\[
Q_0 v(1 - n)(1 + \gamma) + \gamma + v^2 = n(1 + \gamma) = 0 ,
\]
\[
N_0' = -\frac{\tilde{e}}{2} \left( \frac{2y + 1 + v^2}{1 - v^2} \right),
\]
\[
Q_0 = \frac{\tilde{e}}{2} (1 + \gamma) \left[ \frac{Q_0(1 + v^2) + 2v}{1 - v^2} \right],
\]
\[
\tilde{e}' = -2 \cot \Theta_0 .
\]

We may give equation (43) the following quasi-Newtonian interpretation: because of nonzero pressure, the effective gravitational mass-energy is enhanced by the factor \((1 + \gamma)\) for both the effects of gravitation \(n\) and the dragging of inertial frames \(Q_0 v(1 - n)\); the net effect of these two terms is balanced per unit inertial mass energy by the "centrifugal term" \(v^2\) (with the "radius" scaled out in this self-similar problem) and the pressure term \(\gamma\). Similarly, equation (44) is the analog of the Newtonian relationship that the vertical gravitational field \((\propto N_0')\) just above the surface of the disk is equal to \(-2\pi G\) times the local surface mass density \((\propto \tilde{e});\) the extra factor represents various relativistic corrections (see the first relation of eq. [41]). The jump condition (eq. [45]) on the derivative of the frame-dragging term \(Q_0\) and the geometrical distortion (eq. [46]) of the angular location of the midplane of the disk have, of course, no Newtonian analogs.

For a cold, slowly rotating disk, where \(\gamma \leq v^2 \ll 1\), the frame-dragging term \(\propto Q_0 v\) is negligible, and equation (43) recovers the Newtonian approximation for a centrifugally supported Mestel disk: \(n \approx v^2\). On the other hand, from equation (49), \(N'\) in this limit has a solution that satisfies \(N' = 0\) at \(\Theta = 0\) given by
\[
N' \approx -\frac{2n}{1 + n} \left( \frac{1 - \cos \Theta}{\sin \Theta} \right),
\]
with equation (46) yielding the location of the disk midplane at \(\Theta_0 \approx \pi/2\). Thus, \(N_0' \approx -2n \approx -2v^2\), and equation (44) now leads to the solution \(\tilde{e} \approx 4v^2\), where \(\tilde{e}\) is itself obtained from the first relation of equation (28) as \(\tilde{e} \approx 8\pi \tilde{\Theta}^2 \tilde{e}\) (see eq. [61]). Thus, we have the identification \(v \approx v^2/(2\pi \tilde{\Theta}^2)\), which corresponds to a Newtonian surface mass density \(\Sigma = c^2 \tilde{\Theta}^2 / G\) (radius \(\tilde{\Theta}\) now having the dimensions of length) related to the disk rotational velocity \(V = cv\) given by the famous Mestel formula:
\[
\Sigma = \frac{V^2}{2\pi G \tilde{\Theta}} .
\]

For the fully relativistic situation, the disks are characterized by values of \(v\) and \(\gamma\) that are not very small compared to unity. In such a situation, one approach could be to specify these two parameters and solve the problem numerically with \(n\) and \(\Theta_0\) as eigenvalues. In practice, such an approach is very costly. We would have to adopt a shooting method in three dimensions for \(n, \Theta_0, \) and \(\eta = Q(0)\). For nonlinear ODEs, the number of operations increases exponentially with the number of eigenvalues. On the other hand, the values of \(v\) and \(\gamma\) do not come into play until we get to the disk because the right-hand sides of equations (37), (38), and (39) vanish when \(\Theta \neq \Theta_0\). Therefore, it is more efficient to treat \(n\) and \(\eta\) as our nominal solution space parameters and solve for \(\Theta_0, v, \gamma, \) and \(\tilde{e}\) from equations (43), (44), (45), and (46) once we have integrated the ODEs (37), (38), and (39) for the properties of spacetime off the disk plane.

The quantities \(\Theta_0, v, \gamma, \) and \(\tilde{e}\) enter in equations (43)–(46) with a pattern that allows us to proceed as follows: The dynamic equations (37), (38), and (39) are cast for \(\Theta = \Theta_0\) as a set of two second-order ODEs in \(N\) and \(Q:\)

\[
N'' = Q^2 \left( \frac{1 - n}{1 + n} \right)^2 + (Q' - Q \cot \Theta + QN')^2
\]
\[
- \frac{2n}{1 + n} - N' \cot \Theta ,
\]
\[
Q'' = Q \left( 1 + Q^2 \left( \frac{1 - n}{1 + n} \right)^2 + (N' - \cot \Theta)^2 \right)
\]
\[
- (Q' - Q \cot \Theta + QN')^2 \right] - Q' \cot \Theta .
\]

For given \(n\) and \(\eta = Q(0)\), we begin with the boundary values \(Q = 0, Q' = \eta, N = 0,\) and \(N' = 0\) at the pole \(\Theta = 0\) and integrate toward the disk at \(\Theta = \Theta_0\) (whose value is unknown at this point). At each potential choice \(\Theta\) for \(\Theta_0\), we solve equation (43) for \(1 + \gamma:\)
\[
1 + \gamma = \frac{1 - v^2}{(Q_0 v + 1)(1 - n)},
\]
where \(v\) is obtained by the following procedure: We first divide equation (45) by equation (44):
\[
A = -\frac{Q_0}{N_0'} = \frac{(1 + \gamma)[Q_0(1 + v^2) + 2v]}{2(1 + \gamma) - (1 - v^2)} ,
\]
with the value of \(A\) known from the off-plane integration to the candidate \(\Theta\) for \(\Theta_0\). With the elimination of \(1 + \gamma\) through equation (51), the last equation implies
\[
Q_0 v^2 + [2 + AQ_0(1 - n)] v + Q_0 - A(1 + n) = 0 .
\]

This quadratic equation for \(v\) yields a solution,
\[
v = -\frac{1}{Q_0} \frac{A}{2} (1 - n)
\]
\[
+ \sqrt{\frac{1}{Q_0} + \frac{A}{2} (1 - n)^2} - 1 + \frac{A}{Q_0} (1 + n) ,
\]
where we have chosen the sign so that \(v\) is well behaved, going to \((1 + n)A/2 \to 0\), when \(Q_0 \to 0\).

Once \(v\) and \(\gamma\) are known, we can calculate the rescaled energy density,
\[
\tilde{e} = \frac{2(1 - v^2)N_0'}{1 + 2\gamma + v^2} .
\]

We have found the disk when \(\Theta\) has a value \(\Theta_0\) such that equation (46) holds:
\[
\cot \Theta_0 = -\frac{1}{2} \gamma \tilde{e} .
\]
Since the reduced energy density \(\tilde{e}\) is positive, we obtain a disk location \(\pi/2 < \Theta_0 < \pi\). With the other obvious limits,
\[
0 < v < 1 , \quad 0 < \gamma < 1 ,
\]
our parameter space is confined to
\[
0 < n < 1 , \quad \eta > 0 .
\]
5. RESULTS

Contours of constant $\gamma$ in $n-v$ space are plotted in Figure 1. Roughly speaking, $n$ is a measure of the strength of the gravitational field. It ranges from $n = 0$ for flat space to $n = 1$ for maximum rotation. One may also argue that since our similarity transformation is $r \to ar$, $t \to a^{(1-n)/2}$, $n$ cannot exceed unity or the passage of time would proceed more quickly deeper in the gravitational potential well, contrary to common experience in general relativity. Hence, $n$ lying within the interval $(0, 1)$ is an anticipatable result from the self-similarity of the basic problem. From Figure 1 we also see that equilibria require lower $v$ for given $n$ if $\gamma$ is larger; this result conforms with the Newtonian intuition that less gravity if there is a greater degree of pressure support.

For each $\gamma$, there is a maximum rotational velocity $v_c$ above which there is no equilibrium. Table 1 gives the numerical value of $v_c$ as a function of the parameter $\gamma$. Our computed value $v_c = 0.436$ when $\gamma = 0.004$ is consistent, if we perform a simple extrapolation, with the estimate of Lynden-Bell and Pineault (1978b), $v_c = 0.438$ when $\gamma = 0$, and the latter numbers are what are entered as the first entry of Table 1.

![Figure 1](image1)

**Fig. 1.—Gravity index $n$ vs. rotational velocity $v$ for different values of the pressure parameter $\gamma$. The dashed line is the empirical approximation to $n$ for $\gamma = 0$, found by Lynden-Bell & Pineault.**

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</table>

**Table 1.**

**Critical Velocity as a Function of Sound Speed Squared**

To understand the last result physically, we examine in Figure 2, for various choices of $\gamma$, the behavior of the term $Q_0$ that governs the dragging of inertial frames in the disk plane. Nonzero values of $Q_0$ represent how hard one must accelerate to remain at constant $\phi$. In our metric, $g_{tt} \propto (1 - Q^2)$. Thus, whenever $Q > 1$, an ergoregion develops, and the timelike Killing vector $\partial/\partial t$ becomes spacelike. Since $Q$ only has dependence on $\theta$, the ergoregion is best described by the exterior of a cone whose opening angle $\theta_{\text{ergo}}$ is defined by $Q(\theta_{\text{ergo}}) = 1$. Naturally, the ergoregion first appears on the disk when $Q_0 = 1$ and $\theta_{\text{ergo}} = \theta_0$. If we assume that $Q$ is continuous and monotonic, then $\theta_{\text{ergo}}$ decreases (for the ergoregion above the disk) as $Q_0$ increases until finally $\theta_{\text{ergo}} \to 0$ as $Q_0 \to \infty$. In this limit, when the ergoregion occupies the entire space above (and below) the disk, equation (43) may be balanced only when $n \to 1$, which recovers the upper bound on $n$. The rotation velocity at which $Q_0$ diverges is then determined by a limit process on the product $Q_0(1 - n)$.

5.1. Surface Density of Models

Table 2 gives the dimensionless coefficient

$$\bar{\Sigma} \equiv \frac{(1 + n)\bar{\Sigma}}{8\pi},$$

(multiplied by 100 to avoid writing too many zeroes) corresponding to a given pair of values $v/v_c$ and $\gamma$ that characterize an equilibrium model. Notice that for small $\gamma \ll (v/v_c)^2 \ll 1$, we recover the Mestel solution, $\bar{\Sigma} \approx v^2/2\pi$. In principle, if one takes the attitude that gas pressures must be three-dimensionally isotropic rather than two-dimensionally, as idealized in this paper, then disks cannot remain vertically thin unless $\gamma \ll v^2$. In these restricted circumstances, the entries in Table 2 that violate this constraint are not physically self-consistent. Under a broader interpretation of what might be acceptable in the physical world, relativistic singular isothermal disks (SIDs) might be constructed from noninteracting dark matter particles, in which case allowable stress tensors include diagonal forms that are nonisotropic in the sense of equation (20). In the Newtonian regime, it is also known that strongly magne-
TABLE 2

100 $\delta$ AS A FUNCTION OF $v/v_c$ AND $\gamma$

<table>
<thead>
<tr>
<th>$v/v_c$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
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<tr>
<td>$\gamma$</td>
<td>0.0</td>
<td>0.0313</td>
<td>0.124</td>
<td>0.280</td>
<td>0.498</td>
<td>0.780</td>
<td>1.13</td>
<td>1.53</td>
<td>2.00</td>
<td>2.54</td>
<td>3.14</td>
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<td>1.24</td>
<td>1.33</td>
<td>1.48</td>
<td>1.67</td>
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<td>2.73</td>
<td>3.42</td>
<td>4.40</td>
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<td>1.99</td>
<td>2.07</td>
<td>2.19</td>
<td>2.36</td>
<td>2.58</td>
<td>2.86</td>
<td>3.28</td>
<td>3.84</td>
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<td>0.3</td>
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<td>2.49</td>
<td>2.55</td>
<td>2.66</td>
<td>2.80</td>
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<td>3.60</td>
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<td>2.89</td>
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<td>3.91</td>
<td>4.05</td>
<td>4.22</td>
<td>4.42</td>
</tr>
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</table>

5.2. Circular Orbits: Existence and Stability

We anticipate that many of the more slowly rotating members of the SIDs studied here are unstable to inside-out gravitational collapse in a similar way as their Newtonian counterparts (Li & Shu 1997; Shu et al. 2000). We leave for a future endeavor the study of the dynamical, self-gravitating stability of relativistic SIDs and the consequent formation of black holes at their centers if they undergo gravitational collapse. Here we ask the simpler question: are relativistic SIDs kinematically stable in the sense of having stable circular orbits for (noninteracting dark matter) test particles of nonzero rest mass? The question is nontrivial because circular orbits of arbitrary sizes around point masses in the Newtonian case are all stable, yet circular orbits lose their stability if they approach too closely the event horizons of relativistic point masses (Schwarzschild or Kerr black holes). Is there a similar loss of orbit stability when we go from Newtonian SIDs to relativistic SIDs?

For a test particle of mass $m$ in the equatorial plane of the disk, the symmetries of the geometry gives the conserved quantities

$$u_t = -\dot{E} = -E/m, \quad u_\phi = \dot{\phi} = l/m. \quad (63)$$

Since $u^\phi = 0$ for an orbit confined to the disk plane, the geodesic equation takes the simple form

$$\left(\frac{d\tau}{d\tau}\right)^2 = g''(-1 - \dot{E}^2 g^\tau + 2Eg^\phi - \dot{\phi}^2) \equiv -2V(r),$$

where $\tau$ is the proper time of the particle and $V(r)$ is the effective potential of the problem,

$$V(r) \equiv \frac{1}{2}e^{-Z_0} \left[ e^{N_0} - r^{-2}(Ee^\tau - Q_0 l e^{N_0 - P_0})^2 + \dot{\phi}^2 r^{-2} e^{2N_0 - 2P_0} \right], \quad (65)$$

and $N_0, P_0, Q_0,$ and $Z_0$ have simple fixed numerical values when $N, P, Q, \text{and} Z$ are evaluated in the disk plane $\theta = \theta_0$. To have a circular orbit, we need $dr/d\tau = d^2r/d\tau^2 = 0$, which implies $V(r) = V'(r) = 0$. These two conditions define the values of specific energy and angular momentum, $E$ and $l$, needed to yield a circular orbit at radius $r$. A little algebra shows that the required values of $l$ and $E$ for a circular orbit with $r = r_0$ are

$$l = \pm \sqrt{F - l e^{N_0 - 2P_0} r_0}, \quad E = (\sqrt{F} \pm Q_0 \sqrt{F - 1}) e^{N_0/2 P_0^2}, \quad (66)$$

where we have defined

$$F \equiv \frac{2(1 - n) - Q_0^2}{2(1 - Q_0^2)} \frac{Q_0^2}{4n(1 - n)^2}. \quad (67)$$
In equations (66) and (67), the upper sign choice corresponds to prograde orbits, the lower, to retrograde ones. For the upper sign choice, the numerator in equation (67) goes through zero when the denominator does; i.e., for any allowable \( n \), \( F \) stays positive when the disk becomes an ergoregion as \( Q_0 \) crosses unity. From our earlier discussions, we recognize the factor \( r_0^2 \) as the correct scaling to account for the contribution to \( \tilde{E} \) of the gravitational “potential energy” per unit mass. We shall prove below that \( F > 1 \) and that the term \( \sqrt{F} \) represents the contribution of the rest and kinetic energies. Notice now that frame dragging adds a positive contribution to the specific energy \( \tilde{E} \) of prograde circular orbits, whereas it adds a negative contribution for retrograde circular orbits. Moreover, with the lower sign choice in equation (67), \( F \) diverges to \( +\infty \) when \( Q_0 \to 1^- \); i.e., the two terms in \( \tilde{E} \) cancel to lowest order for large \( F \) when the disk plane first becomes an ergoregion. Massive test particles in retrograde circular motion have zero total energy in this limit.

The physical interpretation of these results follows from examining the three-velocity of circular orbits in the tetrad frame:

\[
v_p = \frac{u^{(0)}}{u_0} = \pm \sqrt{\frac{F - 1}{F}}, \tag{68}
\]

where, again, the upper choice corresponds to prograde motion, the lower, to retrograde motion. Equation (68) shows that \( F \) is the square of the Lorentz factor of the particle motion in the tetrad frame: \( F = 1/(1 - v^2) \). The first term \( \sqrt{1/F} \) in the expression for \( \tilde{E} \) thus represents the usual special relativistic contribution to the rest and kinetic energies of a particle. Moreover, the expression for specific angular momentum \( \tilde{l} \) in equation (66) is now recognized as the velocity \( v_p \) in the \( \phi \)-direction times the usual Lorentz factor correction, times not the radius \( R_0 \) but the circumference of the orbit \( \pi R_0 \), divided by \( 2\pi \). The above interpretation for \( F \) explains why the sequence of retrograde circular orbits terminates when \( Q_0 \to 1^- \); when the disk develops an ergoregion, massive particles in retrograde motion must travel at the speed of light (\( F \to \infty \)) if they are to resist the dragging of inertial frames.

No such difficulty affects particles in prograde circular orbit. Equations (66) and (68) require \( F > 1 \) to make physical sense. We have checked numerically that \( F \) as given by equation (67) exceeds unity for all disks in which the fluid velocity \( v \) of the disk is moderately smaller than \( v_p \) for any value of \( \gamma \). When \( v \) approaches the critical velocity \( v_c \), where the frame-dragging parameter \( Q_0 \) becomes very large, numerical errors prevent us from confirming that prograde circular orbits exist. An analytic argument relieves our worries on this score.

For a fixed value of \( \gamma \), the solution sequence terminates when \( Q_0 \to \infty \). Let us evaluate \( F \) in this limit. The equation of motion of the disk matter (eq. [43]) may be written as

\[
1 - n = \frac{1 - v^2}{(Q_0 v + 1)(1 + \gamma)}. \tag{69}
\]

Thus, when \( Q_0 \to \infty \), equation (67) with the upper sign choice becomes

\[
F \to \frac{1}{2} \left[ 1 + \sqrt{1 + 4v^2(1 + \gamma)^2/(1 - v^2)^2} \right], \tag{70}
\]

which explicitly satisfies \( F > 1 \) for any values of \( v > 0 \) and \( \gamma \geq 0 \). In the same limit and with the same upper sign choice, the three-velocity of the test particle in equation (68) is given by

\[
v_p = \frac{\sqrt{-1 + \sqrt{1 + 4v^2(1 + \gamma)^2/(1 - v^2)^2}}}{1 + \sqrt{1 + 4v^2(1 + \gamma)^2/(1 - v^2)^2}}. \tag{71}
\]

Notice that \( v_p = v \) when \( \gamma = 0 \). In other words, the velocity of a test particle in a prograde circular orbit equals the velocity of the disk matter when the latter has zero pressure—a satisfying consistency check of the result.

We now wish to investigate whether circular orbits are stable. Imagine perturbing the radial position \( \ell \) of the test particle about its equilibrium position \( r_0 \), by a small amount \( \ell_1 \), keeping \( \tilde{E} \) and \( \tilde{l} \) fixed. To lowest nonvanishing order on expansion about \( r_0 \), equation (64) becomes

\[
\left( \frac{dr_1}{dt} \right)^2 = -V''(r_0)r_1^2. \tag{72}
\]

Stability of the motion depends on the sign of \( V''(r_0) \). If \( V''(r_0) > 0 \), then circular orbits are stable because there are no perturbations—maintaining the same specific energy and angular momentum—that will produce a real solution for \( r_1 \) in the above equation. In this case, of all the orbits of a given specific angular momentum \( \tilde{l} \), the circular orbit has the least specific energy; therefore, it is not possible to perturb the circular orbit from its equilibrium without giving the particle some additional specific energy, which will then cause it to oscillate about the equilibrium radius \( Q_0 \) with an “epicyclic” frequency \( |V''(r_0)|^{1/2} \). The radial motion may be pictured as rolling up and down the walls of a “valley.” On the other hand, if \( V''(r_0) < 0 \), then circular orbits are unstable because a small perturbation of such an orbit—even one that retains the original specific energy and angular momentum—will lead to exponentially growing departures from the equilibrium radius \( r_0 \). In this case, of all the orbits of a given specific angular momentum \( \tilde{l} \), the circular orbit has a (local) maximum of specific energy, and the test particle becomes unstable by rolling off a “hill.”

In detail, after some algebra, we obtain

\[
V''(r_0) = (1 - n)e^{\lambda}r_0^2\left[ n(\sqrt{\tilde{F}} \pm Q_0\sqrt{F - 1})^2 + (1 - Q_0^2(F - 1)) \right]. \tag{73}
\]

where we have made use of the expressions for \( \tilde{l} \) and \( \tilde{E} \) from equation (66). When \( Q_0 = 0 \) (and hence \( v = 0 \), \( 1 - n = 1/(1 + \gamma) \)), \( F = 1 + \gamma \), and \( V''(r_0) > 0 \). In this limit, retrograde and prograde circular orbits are both stable. In the limit \( Q_0 \to 1^- \), \( F \to 1/(1 - n^2) \) for prograde orbits and \( F(1 - Q_0^2) \to (1 + n)/(1 - n) \) for retrograde orbits. Thus, \( V''(r_0) \) remains positive for both cases. Retrograde and prograde circular orbits are still stable forms of motion at the onset of the disk’s development of an ergoregion, although even the most rapidly moving retrograde particle finds it difficult to resist frame-dragging when \( Q_0 \to 1^- \). As \( Q_0 \) passes through unity and approaches \( \infty \), retrograde circular motion at velocities less than the speed of light becomes impossible, but the product \( (1 - n)Q_0 \to (1 - v^2)/(1 + \gamma)v \) remains finite and positive, so \( V''(r_0) > 0 \) for prograde circular orbits. We have verified numerically that \( V''(r_0) \) stays positive between these various limits. In summary, prograde
circular orbits exist and are stable for power-law isothermal disks from the nonrelativistic to the ultrarelativistic regime, whereas retrograde circular orbits are possible and stable only for disks that do not develop an ergoregion.

We may state the result in an alternative way that relates to known results concerning circular orbits around black holes. Of all orbits of a given specific energy, the circular orbit in a stable/unstable situation has a (local) maximum/minimum of specific angular momentum. In the case of a Schwarzschild black hole, we know that circular orbits that start off at great distances from the event horizon, \( r_0 \gg r_{\text{Sch}} \), are close to the Newtonian limit and therefore are stable. They remain stable as long as the specific angular momentum of the circular orbit continues to decrease with decreasing orbital radius (or circumference). There comes a point, \( r_0 = 3r_{\text{Sch}} \), when the square of the “epicyclic frequency” \( V''(r_0) \) changes sign, and the specific angular momentum \( I \) of the circular orbit has an inflection point and starts to increase for decreasing \( r_0 \). This violation of “Rayleigh’s criterion” signals a transition from stable to unstable circular orbits. Because of frame dragging, the case of Kerr black holes is more complicated but can be similarly elucidated, as we have done above for the disk case.

In our power-law disks, every radius \( r_0 \) is similar to any other radius, and spacetime is not flat even at infinity. Thus, if a circular orbit is stable at any radius in a given model, circular orbits at all radii are stable. It is hard to imagine how one could realistically construct a self-gravitating disk of rotating material otherwise.

6. PHOTON ORBITS

The case of photon orbits in the spacetime of relativistic SIDs is also interesting. Our self-similar metric equation (3) admits a homothetic Killing \( \xi \) satisfying

\[
\mathcal{L}_\xi g_{\mu\nu} = 2 \xi_{(\mu; \nu)} = 2 g_{\mu\nu} .
\]

In component form, it reads

\[
g_{\mu\nu, x} \xi^x + g_{\mu x} \xi_x \nu + g_{\nu x} \xi_x \mu = 2 g_{\mu\nu} .
\]

The solution to this equation is

\[
\xi^x = [(1 - n)t, 0, r, 0] .
\]

Associated with this vector is a conserved quantity \( \Gamma = \xi^x k^x \) for null geodesics \( k^x = dx^x/d\lambda \), where \( \lambda \) is an affine parameter. Indeed,

\[
\frac{d\Gamma}{d\lambda} = \Gamma ; x k^x = (\xi^x k^x) ; k^x = \xi_{(\mu; \nu)} k^\mu k^\nu
\]

\[
= g_{\mu\nu} k^\mu k^\nu = -m^2 ,
\]

where we have used the geodesic equation \( k^\mu ; \mu = 0 \). Therefore, for a massless particle, \( m = 0 \), and \( \Gamma \) is a constant of motion.

In addition, we have the two ordinary Killing vectors associated with the stationarity and axial symmetry of the spacetime:

\[
\xi^{(t)} = \frac{\partial}{\partial t} , \quad \xi^{(\phi)} = \frac{\partial}{\partial \phi} .
\]

In total, we have the following three conserved quantities:

\[
E = -k_t , \quad l = k_\phi ,
\]

\[
\Gamma = (1 - n)tk_x + r_k_x \Rightarrow k_x = \frac{1}{r} \left[ \Gamma + (1 - n)tE \right] .
\]

The null condition \( k^x k_x = 0 \) can be used to determine \( k_\phi \):

\[
k_t^2 g^{tt} + 2k_t k_\phi g^{\phi t} + k_\phi^2 g^{\phi\phi} + k_r^2 g^{rr} + k^\mu g^{\mu\nu} = 0 ,
\]

which implies

\[
k_\phi = \pm \frac{1}{\sqrt{g_{\phi\phi}}} \left\{ -E^2 g^{tt} + 2Erg^{\phi t} - I^2 g^{\phi\phi} \right. \\
- \left. \frac{1}{r^2} \left[ \Gamma + (1 - n)tE \right]^2 g^{rr} \right\}^{1/2} \nonumber \\
= \pm e^{2/2 \left[ E_r^1 - \nu - N - Q l e^{-P} \right] - I^2 e^{-2P} } \nonumber \\
- \left[ \Gamma + (1 - n)E_t \right]^2 e^{-2} \right\}^{1/2} .
\]

Finally, the geodesic is described by

\[
k_t' = Er^{-2} e^{-N} - r^{-1} + n e^{-P} l \nonumber \\
k_\phi' = r^{-1} + n e^{-P} l \nonumber \\
k_r' = e^{N} l q^{-1} \left[ \Gamma + (1 - n)E_t \right] \nonumber \\
k_\phi' = \pm r^{-1} e^{N} l q^{-1} \left[ E_r^1 - \nu - N - Q l e^{-P} \right] \nonumber \\
- l^2 2 e^{-2P} - \left[ \Gamma + (1 - n)E_t \right]^2 e^{-2} \right\}^{1/2} .
\]

Divide everything by \( k_t' \), and we have

\[
\frac{d\phi}{dt} = \frac{Q + \alpha(1 - Q^2)}{1 - Q\alpha} r^{n-1} e^{N-P} ,
\]

\[
\frac{d\ln r}{dt} = \frac{e^{2N-2P} \beta}{1 - Q\alpha} ,
\]

\[
\frac{d\Theta}{dt} = \pm \frac{e^{N-Z-1}(1 + n) \alpha}{1 - Q\alpha} \nonumber \\
\times \left[ (1 - Qx)^2 - \alpha^2 - \beta^2 e^{2N-Z} \right]^{1/2} ,
\]

where

\[
\alpha \equiv \frac{l}{E} r^{n-1} e^{N-P} , \quad \beta \equiv \left[ \frac{\Gamma}{E} + (1 - n) \right] r^{n-1} .
\]

To make this system autonomous, we extend the space to include \( x \) and \( \beta \) as two of the variables. The self-similarity of the problem makes it convenient to define the reduced radius and time as

\[
\zeta \equiv \ln r \quad \text{and} \quad \delta t = r^{n-1} dt .
\]
6.1. Dynamics and Geometry of the Photon Trajectories

We need to specify initial values for the six dependent variables \((\phi, \zeta, \Theta, \beta, \alpha, t)\) and integrate forward in \(\tau\). Without loss of generality, stationarity, axial symmetry, and self-similarity imply that we may take \(t = 0, \phi = 0,\) and \(r = 1\) (or \(\zeta = 0\)) at \(\tau = 0\). We shall also assume that all photons begin by being emitted from the plane of the disk, \(\Theta = \Theta_0\). Except for a set of measure zero (involving photons traveling outward exactly along the rotation axis), we shall find that this last assumption also results in no loss of generality because even photons emitted by external sources outside of the disk are soon bent to cross the disk plane. This behavior can be attributed to the existence of an adiabatic invariant \(J\) (see below), which places a constraint on the trajectories different from those presented by the classical integrals \(l, E,\) and \(\Gamma\). In any case, we are now left with only two arbitrary specifications, the initial values \(\alpha = \alpha_\ast\) and \(\beta = \beta_\ast\), from which equation (89) allow us to reconstruct the two constants of motion,

\[
\frac{l}{E} = \alpha_\ast, \quad \Gamma/E = \beta_\ast.
\]

6.2. Sign Choices

To solve the geodesic equations (91)–(96) numerically, we need to be careful about the plus/minus signs and the term
in the square root of equation (93), which we call $\Lambda$:
\[
\Lambda \equiv (1 - Q\alpha)^2 - \beta^2 e^{2N - z} .
\] (110)

In order for $d\Theta/d\tau$ to be real, we need $d\Lambda/d\tau$ to vanish whenever $\Lambda$ attains zero from positive values (otherwise $\Lambda$ can become negative). By straightforward differentiation, it is easy to show
\[
-\frac{1}{2} \frac{d\Lambda}{d\tau} = (Q - Q^2 \alpha - \alpha^2) \frac{d\alpha}{d\tau} + \beta \frac{d\beta}{d\tau} e^{2N - z} ,
\] (111)

where we have used that $d\Theta/d\tau = 0$ when $\Lambda = 0$ to eliminate derivatives of functions of only $\Theta$. On substitution of equations (94) and (95), the last expression becomes

\[
[Q + \alpha(1 - Q^2)]\alpha(n - 1) \frac{e^{2N - z\beta}}{1 - Q\alpha} + \beta e^{2N - z} 
\times (1 - n)\left[1 - \frac{e^{2N - z\beta}}{1 - Q\alpha}\right] .
\] (112)

With $\Lambda = 0$ in equation (110), the term in the last square bracket equals

\[
\left[\frac{(1 - Q\alpha)Q\alpha + \alpha^2}{1 - Q\alpha}\right] ,
\] (113)

and when this is substituted into equation (112), we find that the first and second terms algebraically cancel. Thus, $d\Lambda/d\tau$ vanishes when $\Lambda = 0$, which is the desired result.

Initially, we require that for $\psi < \pi/2$, $d\Theta/d\tau$ is negative:
\[
\frac{d\Theta}{d\tau} = -(1 + n) e^{N - z/2} \cos \psi .
\] (114)

Similarly, we want $d\zeta/d\tau$ positive whenever $-\pi/2 < \chi < \pi/2$. This requires, after a little algebra,
\[
\frac{d\zeta}{d\tau} = \pm e^{N - z/2} \sin \psi \cos \chi > 0 ;
\] (115)
i.e., we choose the plus sign in the initial conditions for $\alpha$ and $\beta$. We can then integrate $\Theta$ forward in $\tau$ until $\Lambda = 0$, where we change the sign of $d\Theta/d\tau$ (and correspondingly the sign in $d\alpha/d\tau$).

6.3. Results

First we shall discuss the photon orbits in a spacetime without the ergoregion ($Q_0 < 1$). As an example, we choose the parameters $n = 0.4$ and $\gamma = 0.5$ (or equivalently, $v = 0.16$) and consider photons launched in different directions as seen by the LNRO. We assume that photons cross the disk plane without absorption or scattering, an assumption that is more likely to apply to massless low-energy neutrinos than real photons, for which a disk with surface density $\sim 10^{23} \text{g cm}^{-2}$ (if we are talking about stellar mass disks of sizes $\sim 1 \text{km}$) is not likely to be optically thin. Our formal usage of the phrase “photon orbits” must henceforth be understood as either a metamorphical rather than literal device or applicable only while the photon is traveling above or below the disk plane.

We find that outgoing photons with $-\pi/2 < \chi < \pi/2$ spiral out to $\zeta = \infty$, while ingoing photons with $\chi > \pi/2$ or $\chi < -\pi/2$ reach a minimum radius and then spiral out to $\zeta = \infty$ again. Only one photon launched along the disk with $\psi = \pi/2$ and $\alpha = 0$ reaches the origin. This was first discovered by Lynden-Bell & Pineault (1978b) for the cold disk. Photons launched in the retrograde direction reach a minimum coordinate angle $\phi_{\text{min}}$ and are then dragged to go in the forward direction. Figure 3 shows these orbits.

We now discuss the out-of-plane behavior of the orbits. Figure 4 shows a typical photon trajectory. No matter what the initial condition is, almost all orbits are focused and eventually trapped by the disk. A (noninteracting) photon will typically penetrate the disk many times. Each time it reaches a turning point $\Theta_i$, where $i$ labels the number of penetration (see Fig. 6). The only trajectory that can escape falling into the disk is one launched exactly along the rotation axis. However, such a photon suffers an infinite redshift as it propagates away from the disk, and an observer located at an infinite distance above or below the disk cannot see it. (This result was first discovered for counter-rotating disks and is probably generic to self-similar configurations that contain an infinite total mass.) Figure 5 exhibits the behavior that a slight deviation from the symmetry axis leads to the confinement of the photon by the disk.

When interactions with the disk are ignored, it appears that $\Theta_i$ decreases with each disk crossing $i$. The result can be
demonstrated to arise from adiabatic invariance. The relevant invariant is easily computed conceptually. The conjugate momentum to $\theta$ is $k_\theta$. Hence, we can define the action integral,

$$J \equiv \oint k_\theta d\theta = \frac{2}{1 + n} |E| \int_{\Theta_{\min}}^{\Theta_{\max}} r^{1-n} \times \left[ (1 - Q\alpha)^2 e^{Z^2 - 2N} - \alpha^2 e^{Z^2 - 2N} - \beta^2 \right]^{1/2} d\Theta.$$  (116)

Here we may treat $\alpha$ and $\beta$ as functions of $\Theta$ along the photon's trajectory during the current cycle. The quantity $r(\Theta)$ varies slowly over this one cycle because it is a monotonic function that does not oscillate. Hence, we may approximate it by its mean value. To zeroth order, we may take $r(\Theta)$ outside of the integral. Thus,

$$J = |E| r^{1-n} [\mathcal{J}(\Theta_{\max}) - \mathcal{J}(\Theta_{\min})],$$  (117)

where

$$\mathcal{J}(...) \equiv \frac{2}{1 + n} \times \int_{0}^{\Theta_{m}} \left[ (1 - Q\alpha)^2 e^{Z^2 - 2N} - \alpha^2 e^{Z^2 - 2N} - \beta^2 \right]^{1/2} d\Theta.$$  (118)

Because the integrand is a known (numerical) function of $\Theta$, we can tabulate the integral $\mathcal{J}$ as a function of its argument $\Theta_m$ for each photon trajectory. We can then invert the expression equation (117) to recover $\zeta = \ln r$ as a function of $\Theta_{\min}$ and $\Theta_{\max}$:

$$\zeta = -\frac{1}{1 - n} \ln \left[ \mathcal{J}(\Theta_{\max}) - \mathcal{J}(\Theta_{\min}) \right] + C,$$  (119)

where $C$ is the constant $\ln(J/E)$ divided by $1 - n$. The resulting curve is plotted as a dashed locus in Figure 6. The concordance between the dashed curve and the actual envelope of the solid photon trajectory is a measure of the goodness of the adiabatic invariant $J$.

It might be argued that the validity of the focusing effect on photons seen in Figure 6 is compromised by the assumption that photons continue in their original direction when they cross the disk plane. However, since all orbits, except for a set of measure zero, always return to the disk plane, the qualitative effect remains the same even if we were to allow photons to interact with the disk matter. The absorption and reemission or the scattering of photons when they cross the disk would result in a slow transfer of photons from the inner disk to the outer disk because no photon can permanently escape from the disk. The focusing of photons toward the equatorial plane is perhaps a generic feature of relativistic disks (and not just peculiar to these self-similar configurations) and may present an obstacle to some classes of beamed-jet models for gamma-ray burst sources.

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**Fig. 4.** Typical photon trajectory. Here the distance is plotted on a log scale, normalized such that the closest approach to the origin is 1. The nonoscillatory curve at the bottom is the projection of the photon trajectory onto the equatorial plane of the disk.

**Fig. 5.** Photon launched along $\Theta = 10^{-6}$ being trapped by the disk. Again, the distance is plotted on a normalized log scale.

**Fig. 6.** Photon orbit showing $\Theta$ as a function of $\zeta$. The turning point in $\Theta$ decreases each time after the photon penetrates the disk. The solid line is from solving geodesic equation directly, while the dotted line is the computed envelope from adiabatic invariance.
Fig. 7.—Behavior of photon trajectories launched in the forward direction with an ergoregion.

Photon orbits in a spacetime with $Q_0 > 1$ are more complicated than those discussed so far. As an example, let us take $n = 0.75$ and $v = 0.32$ (and $\gamma = 0.5$). All forwardly propagating photons ($\chi > 0$) escape to $\zeta = \infty$. The ones launched outward ($\chi < \pi/2$) escape directly, with the $\theta$ dependence mimicking the behavior of those in spacetime without an ergoregion. The ones launched inward ($\chi > \pi/2$) reach a minimum radius and then escape. In the interior region, the photons spiral toward the axis above and below the disk. The turning points in $\theta$ increase as the photons approach the minimum radius. Once past the starting point $\zeta = \ln r = 0$, we see the familiar $\theta$ behavior governed by the adiabatic invariance described above. Figure 7 shows the $\zeta$ and $\theta$ behavior of these trajectories.

The backward photons are divided into two classes, separated by the surface $E_0$. Roughly speaking, this surface corresponds to $E = 0$ in the outgoing direction. All ingoing photons fall toward the origin directly. For an outgoing photon, if $\chi > \chi_c$, then it escapes to infinity. On the other hand if $\chi < \chi_c$, the photon reaches a maximum radius and then falls to the origin (see Fig. 8). For the spacetime considered here, the $E = 0$ surface is plotted in Figure 9. Lynden-Bell & Pineault (1978b) gave an analytic expression for $\chi_c(\pi/2) = \sin^{-1}(-1/Q_0)$. For this particular spacetime, $Q_0 \approx 4.41$. Therefore, $\chi_c(\pi/2) = -0.073$, which agrees with our numerical result.

### 7. Conclusion

We have solved by semianalytic means the Einstein field equations for axisymmetric, self-similar, relativistic disks with “flat” rotation curves, including finite levels of pressure support. These spacetimes are not asymptotically flat and cannot describe correctly the behavior of isolated astrophysical objects when examined at distances that are very large compared to their natural gravitational radii. Nevertheless, the solutions may yield some insight into the near-field solutions of rapidly rotating, compact objects.

As expected from first principles, the solution space is parameterized by two dimensionless numbers, $v$ and $\gamma$, that represent the disk rotation speed and the square of the isothermal sound speed, respectively, when both are normalized appropriately by the speed of light $c$. The qualitative behavior of these disks resembles those found by Lynden-Bell & Pineault (1978b) for the cold disk. This is encouraging because cold disks are known in their Newtonian limits to be violently unstable to a wide variety of...
spiral and barlike perturbations, and we cannot expect their relativistic counterparts to behave much better. A proper stability analysis of the disks discussed in the current paper remains a task for the future.

Ergoregions develop for relatively low rotation velocities in our disks and take the shape of (the outside of) a cone centered around the axis. As the rotational velocity increases, the “ergocone” closes up toward the axis. For each $\gamma$, there is a maximum velocity $v_c$ beyond which no equilibrium can exist because of infinite frame dragging. It should be noted that this maximum velocity lies well below the special relativistic limit of the speed of light.

We examined the behavior of test particles with nonzero rest mass placed in circular orbits in the plane of the disk. We found that prograde circular orbits exist and are stable for the full range of disk models in this paper. Retrograde circular orbits are also stable when they exist but cannot be maintained against frame dragging by particle velocities less than the speed of light when the disks develop ergoregions.

We also carried out a systematic study of planar and nonplanar photon orbits. Most interestingly, we found that all photon orbits are ultimately attracted toward the plane of the disk because of the operation of a general adiabatic invariant. Although the formal result depends on the disk being optically thin to the propagating photons (an unlikely state of affairs), we gave physical arguments why the generic effect may pose defocusing difficulties for some classes of models of gamma-ray burst sources that rely on beamed jets along the rotation axis of rapidly rotating compact objects. To be sure, the effect in realistic flattened systems that do not have infinite mass and spatial extent may be less dramatic than the one found here for relativistic SIDs. A lower bound on the effect might be obtained by examining the analogous properties of photon orbits in a Kerr geometry, which in other respects mimics the spacetime analyzed in the current paper.

It is our belief that the current investigation has just begun to scratch the surface of a potentially very rich mine for general relativity to explore. As discussed in § 1, the study of self-similar (Mestel) disks in the Newtonian limit has uncovered rich veins relating to the stability and collapse of such objects that have illuminated astronomers’ understanding of real-world objects such as protoplanetary disks and triaxial and spiral galaxies. In addition to serving as useful test beds for numerical relativity codes, the relativistic generalization of such studies could shed light on topics such as the efficiency of gravitational radiation and the possible generation of naked singularities during gravitational collapse.

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