THE EVOLUTION OF PROTOSTARS. I. GLOBAL FORMULATION AND RESULTS

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Received 1980 February 4; accepted 1980 April 3

ABSTRACT

We review the controversy and give a new formulation for the problem of the evolution of protostars. Our method entails the division of the global problem into a set of more manageable subproblems. We derive the jump conditions of the radiative accretion shock which joins the hydrostatic mass-gaining core to the dynamic inner cloud envelope. Different approximations hold with high accuracy in the regions that we call the dust envelope, the opacity gap, the radiative precursor, the accretion shock, and the hydrostatic core. In no region do we need to solve equations more complicated than ordinary differential equations. Thus, standard integration schemes yield the high accuracy needed to resolve complex spatial structures which span many orders of magnitude in density and temperature. Our 1 $M_\odot$ protostar ends its main accretion phase moderately high up in the H-R diagram. This star begins its pre-main-sequence phase of quasi-static contraction on a convective Hayashi track.

Subject headings: stars: evolution — stars: formation — stars: interiors

I. INTRODUCTION

a) Historical Review

The study of newly formed stars remains an active area of astronomical research (see the reviews of Hayashi 1966; Bodenheimer 1972; Larson 1973, 1977; Strom, Strom, and Grasdalen 1975; Woodward 1978). The theoretical foundations were laid by Hayashi and his co-workers (Hayashi, Hoshi, and Sugimoto 1962), who determined the properties of pre-main-sequence stars that derive their luminosities by quasi-static contraction at fixed mass. Above a certain luminosity, a pre-main-sequence star of sufficiently low mass is wholly convective and descends the H-R diagram in an almost vertical path. At the bottom of this convective track, a radiative core develops, and the path turns to join one of the radiative sequences first computed by Henyey, LeLever, and Levee (1955). Later studies pointed out the role of various thermonuclear processes which precede the star's reaching the main sequence (Ezer and Cameron 1962; Iben 1965; Graboske and Grossman 1971).

Hayashi's calculations did not extend to the conditions of very low density and temperature typical of the interstellar matter from which stars are ultimately born. The possibility that protostars in these earlier stages might also be hydrostatic was quickly ruled out. Gaustad (1963; see also Cameron 1962 and Gould 1964) showed that, over a wide range of density and temperature, a contracting cloud fragment would lose thermal energy too fast to allow it to maintain mechanical equilibrium. Thus, a protostar's early history must be characterized by rapid dynamical collapse.

A stellar mass of interstellar matter containing dust grains becomes opaque to the infrared radiation generated by its collapse at a density of about $10^{-13}$ g cm$^{-3}$. Hayashi and Nakano (1965) and Narita, Nakano, and Hayashi (1970) attempted to bypass a detailed calculation of the earlier phases by considering a freely falling polytrope of mass $M$ whose evolution they began to follow only after such mean densities had been reached. Negligible energy is radiated away under such circumstances, and a simple energy budget argument (Hayashi 1966) suffices to show that such a protostar would join the quasi-static pre-main-sequence contraction track at a radius of about 50 $(M/M_\odot)R_\odot$.

Unfortunately, hydrodynamic calculations by McNally (1964) and by Bodenheimer and Sweigart (1968) showed that the gravitational collapse of a non-rotating and nonmagnetic protostellar cloud with initial conditions given roughly by the Jeans criterion would be highly nonhomologous. A runaway increase of the density of the central regions occurred which would be arrested only after the formation of an optically thick and hydrostatic central core. The complete problem—
the initiation of collapse in a 1 $M_\odot$ spherical cloud fragment of initial mean density $\sim 10^{-19}$ g cm$^{-3}$ and temperature $\sim 10$ K, the formation of a hydrostatic central core, and its subsequent accretion of an infalling envelope of gas and dust—was first treated by Larson (1969). Larson found that the accretion phase ends with a central star of 1 $M_\odot$ joining the radiative portion of the pre-main-sequence track at a radius of about 2 $R_\odot$ (see also Larson 1972).

Subsequent investigations, starting with initial conditions similar to those of Larson, have given rise to considerable controversy (Appenzeller and Tschamnuter 1975; Westbrook and Tarter 1975). All workers agree on the inevitability of a core/envelope structure, but other important results have diverged greatly. For example, the final radii and luminosities for a 1 $M_\odot$ protostar have differed by about two orders of magnitude. These discrepancies have been attributed to a number of causes. Hayashi (1970) criticized Larson's treatment of the radiative accretion shock, while Larson (1973) emphasized the importance of the time scale of core accretion (see also Winkler and Newman 1980a, b; and Appendix). Appenzeller (1975) cited uncertain low-temperature opacities, while Westbrook and Tarter (1975) blamed poor numerical methods. Clearly, a fresh approach to the theoretical problem would be helpful in settling this controversy. A further motivation for a new study is to assess the significance of the finding of Cohen and Kuhi (1979) that the H-R diagram of observed T Tauri stars disagrees with all of the dynamical tracks published so far.

Viewed from a broader perspective, the controversy extends beyond the field of star formation. Computing the response of the mass-gaining component in mass transfer binaries, Benson (1970) and subsequent workers (Flannery and Ulrich 1977; Neo et al. 1977; Kippenhahn and Meyer-Hofmeister 1977) also obtain rapidly swelling stars when the mass accretion rate becomes very substantial. This conclusion, which superficially supports the results of Westbrook and Tarter, may pose severe problems for conventional models of the evolution of Algol systems and the precursors of X-ray binaries. Although we should keep these broader issues in mind, we shall concentrate on the protostellar problem in the present series of papers.

b) Purpose of this Study

It is doubtful that a refinement of the computational technique alone would provide a decisive and cogent resolution of the issues under debate. What is also needed is a thorough reappraisal of the physical situation. In what follows, therefore, we present a coherent viewpoint of the complete astrophysical problem of protostar formation and evolution. Our formulation possesses the following advantages:

1. Our method divides the whole protostar calculation into more manageable subproblems. This policy ensures greater ease in the physical interpretation of the final answers.

2. Our method yields results with high numerical accuracy, results which are obtained at a fraction of the computing cost of other techniques.

3. Our method has the flexibility to deal with more issues than just protostar evolution. Problems of spherical accretion onto a variety of stellar objects with a large range in mass accretion rates have found a wealth of recent astrophysical applications. Many of these problems should prove amenable to our approach.

In this paper, we supply an overview of our formulation, together with some significant numerical results obtained with it. The detailed basis of our calculations will be presented in Papers II and III, where we will concentrate, respectively, on the reaction of the underlying star and on the gas dynamics and radiative transfer in the accretion flow. Such a separation is very natural since we will show in the present paper that, as long as the matter flow onto the central core is not appreciably retarded by radiative input of energy or momentum, the two subproblems can be effectively decoupled.

II. FORMULATION OF PROBLEM

a) Basic Equations

Consider the collapse of an isolated spherical fragment. The basic dynamical equations for the matter field are the time-dependent equation of mass conservation, radial force equation, and heat equation:

$$\frac{\partial M}{\partial t} + u \frac{\partial M}{\partial r} = 0, \quad \frac{\partial M}{\partial r} = 4\pi r^2 \rho, \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = - \frac{GM}{r^2} - \frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{1}{c} (\kappa_{\text{abs}} + \kappa_{\text{sca}}) F_{\text{rad}}, \quad (2)$$

$$\rho T \left( \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial r} \right) = c \rho (\kappa_{\text{abs}} F_{\text{rad}} - \kappa_{\text{em}} a T^4)$$

$$- \frac{1}{\rho^2} \frac{\partial}{\partial r} \left[ r^2 (F_{\text{conv}} + F_{\text{rad}}) \right] + \rho \varepsilon, \quad (3)$$

where $\partial/\partial t + u \partial/\partial r$ is the substantial derivative $D/Dt$. In equation (3) we have allowed the emission and absorption opacities to be unequal in case the matter and radiation temperatures differ. In equation (2), we have written the radiation energy flux in the rest frame of the fluid as $F_{\text{rad}}'$ to distinguish it from the flux $F_{\text{rad}}$ seen by an inertial observer. The connection between the two is given by (Simon 1963):

$$F_{\text{rad}} = F_{\text{rad}}' + u (E_{\text{rad}}' + P_{\text{rad}}'), \quad (4)$$
where $E_{\text{rad}}'$ and $P_{\text{rad}}'$ are the radiation energy density and pressure in the fluid frame. We distinguish between $F_{\text{rad}}$ and $F_{\text{rad}}'$ because their difference $u(E_{\text{rad}}' + P_{\text{rad}}')$ may be appreciable at large optical depths $\tau$ where $E_{\text{rad}}' = 3P_{\text{rad}}' - 3F_{\text{rad}}'/c$. In contrast, the differences $(E_{\text{rad}} - E_{\text{rad}}')$ and $(P_{\text{rad}} - P_{\text{rad}}')$ equal $2uF_{\text{rad}}'/c^2$ and are always small compared to $E_{\text{rad}}$ and $P_{\text{rad}}$; provided $u/c$ is small. To see this, merely recall that $F_{\text{rad}}'$ has a limiting value of $cE_{\text{rad}}$ even for monodirectional radiation in optically thin regions. Thus, we shall not bother to distinguish between $E_{\text{rad}}$ and $E_{\text{rad}}'$ or between $P_{\text{rad}}$ and $P_{\text{rad}}'$.

In equations (2) and (3), we have taken into account the fact that the scattering of soft photons transfers momentum but not heat to the matter. When convection arises, we calculate the convective energy flux $F_{\text{conv}}'$ by the mixing-length theory (Baker and Temesvary 1966). We have also included the heat flux $F_{\text{cond}}'$ which arises by electron conduction in the partially degenerate central regions of low-mass protostars.

When matter and radiation are in thermal equilibrium, we have adopted the following opacities. For $T > 10^4$ K, we use the Rosseland opacities of Cox and Stewart (1970) for a solar mix. For $10^4$ K $>$ $T$ $>$ 700 K, we use the Rosseland opacities of Alexander (1974), which include a thermodynamic calculation for the effects of molecules and bare silicate grains. For $T < 700$ K, we extend Alexander's tables with an analytic expression for silicate opacity derived from Gilman's (1974) study of the absorption efficiency of olivine. In our problem, appreciable disequilibrium between matter and radiation occurs only inside the accretion shock where our derivation of the total jump conditions does not appeal to detailed opacity laws (§ II A).

In addition, we have considered the contribution of graphite grains to the low-temperature opacities (Gilman 1974). We assume that these grains have a radius $b$, and that they are well coupled both mechanically and thermally to the gas. (The assumption of a single matter temperature $T$ and a single fluid velocity $u$ should be relaxed for the calculation of high-mass protostars; see Yorke and Krueger 1977.) When the graphite grains flow into high-temperature regions, $b$ decreases by thermal evaporation (sublimation) and by chemical corrosion by water vapor (or free oxygen atoms). These two processes result in the destruction equation (Hollenbach 1978):

$$\frac{\partial b}{\partial t} + u \frac{\partial b}{\partial r} = -A \exp\left(-\frac{e_1}{kT}\right) - B \rho T^{1/2} \exp\left(-\frac{e_2}{kT}\right).$$

(5)

The derivation of the above equation, the values of the constants which appear in it, and the formulae for the silicate and graphite opacities will all be given in Paper III. Here, we wish only to mention that our adopted mass fraction of dust particles, when coated with suitably icy mantles, is roughly consistent with the observational value of $\kappa_a = 250$ cm$^2$ g$^{-1}$ at visual wavelengths (derived from 1 mag of visual extinction for each column density of $2 \times 10^{21}$ H atoms cm$^{-2}$; see Savage and Mathis 1979).

To complete the specification of the matter field, we need the LTE equations of state for the gas pressure and specific entropy:

$$P = P(\rho, T) \quad \text{and} \quad s = s(\rho, T),$$

(6)

as well as the nuclear energy generation rate $\epsilon(\rho, T)$. For most of the relevant $\rho$-$T$ plane, we use the equations of state of Eggleton, Faulkner, and Flannery (1973). At lower $\rho$ and $T$, we solve the Saha equations for the dissociation of hydrogen molecules and the ionization of hydrogen and helium atoms. Protostellar collapse spans a great range of physical conditions, rendering the functional form of equation (6) quite complicated. Thus, we found it most convenient computationally to store the equations of state, along with the Cox-Stewart and Alexander opacities, in tabular form on the computer. For the nuclear energy generation rate, we include deuterium burning at a rate given by Fowler, Caughlan, and Zimmerman (1975).

The radiation field affects the gas dynamics via the frequency-integrated energy density $E_{\text{rad}}$, energy flux $F_{\text{rad}}$, and pressure $P_{\text{rad}}$. These three quantities are related to one another through the first two moments of the equation of transfer. For $|u/c| < 1$, the time-dependent form of these moment equations read

$$\frac{\partial E_{\text{rad}}}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_{\text{rad}}) = c \rho (\kappa_{\text{em}} a T^4 - \kappa_{\text{abs}} E_{\text{rad}})$$

$$- \frac{\rho}{c} (\kappa_{\text{abs}} + \kappa_{\text{sc}}) F_{\text{rad}}',$$

(7)

$$\frac{\partial P_{\text{rad}}}{\partial r} + \frac{1}{r} (3P_{\text{rad}} - E_{\text{rad}}) = - \frac{\rho}{c} (\kappa_{\text{abs}} + \kappa_{\text{sc}}) F_{\text{rad}}'.$$

(8)

Equation (8) differs from equation (34) of Castor (1972) in our neglect of terms like $c^{-2}(DF_{\text{rad}}'/Dt)$ which are smaller than the retained terms by $u/c$ in optically thin regions and by $(u/c)$ (photon mean free path/r) in optically thick regions. In this approximation, equation (8) states that the divergence of the radiation stress tensor (i.e., the steady-state change of the momentum of the photons) is due to the transfer of momentum from radiation to matter. Equation (7) states that the energy carried in the photon field changes either because of a net difference of photons emitted and absorbed by matter or because of the work done on matter by the radiation field. Equation (7) can be manipulated into equation (35) of Castor (1972) by suitably using equations (1), (2), (4), and (8).

To complete the specifications of the radiation field, we need a closure relation between $P_{\text{rad}}$ and $E_{\text{rad}}$. In this work, we shall use the method of the variable
Eddington factor (Mihalas 1978):

$$P_{\text{rad}} = f E_{\text{rad}},$$  \hspace{1cm} (9)

where the variation of $f(r, t)$ will be computed according to the rules given in § IIg and Papers II and III. For here, we merely make the following comment. In optically thick regions, the approximations of a nearly isotropic radiation field, $f = 1/3$, and of good thermal coupling between matter and radiation, $E_{\text{rad}} = a T^4$, leads to the “radiation conduction approximation” whereby $F_{\text{rad}}$ is proportional to the gradient of the gas temperature $T$. The radiation conduction approximation is, of course, totally inappropriate in optically thin regions. In such regions, the matter temperature should be obtained from equation (3) as a balance between cooling by the emission of radiation and heating by the absorption of radiation and by compression. For this purpose, it is important to have an Eddington factor which has the correct geometric behavior. (For example, the combination $\kappa c r < 1$ and $f = 1$ leads to a $1/r^2$ decrease of $E_{\text{rad}}$ and $P_{\text{rad}}$ from eq. [8], which would be the correct behavior for optically thin regions far from a central light source.)

b) Computational Philosophy

The seven partial differential equations (1), (2), (3), (5), (7), (8) of first order generally require seven boundary conditions to specify a unique evolutionary history given the initial state. Three of these apply at the center: $M = 0$, $u = 0$, and $F_{\text{rad}} = 0$ at $r = 0$. Three others can be the specifications of $T$, $b$, and $E_{\text{rad}}$ at the outer boundary. The final condition refers to the mechanics of the outer boundary. Two choices are conventional: constant external pressure or constant cloud volume. In the first case, the surface is a free boundary, and its radial location $r_0$ is determined by kinematic condition that it moves with the fluid. In the second case, the constraint on $P$ may be replaced by the condition that $u = 0$ at the fixed radius of the outer boundary. Other choices are possible (Disney 1976). Once the boundary conditions have been set, the solution of equations (1)-(9) can proceed. This has been the approach followed in all previous collapse calculations.

The direct procedure, although straightforward in principle, has turned out to be difficult and time-consuming in practice (for a very careful recent calculation, see Winkler and Newman 1980a, b). The finite differencing of variables which change by many orders of magnitude requires meticulous attention. The problems in obtaining proper spatial and temporal resolution have led to the controversial situation outlined in § I. We take a completely different approach.

The computational problem can be both simplified and clarified if we turn the enormous variations in physical conditions to our advantage. Rather than ask the computer to calculate automatically with the most general equations, we divide the global cloud collapse problem into physically distinct regions. In each region, different simplifying assumptions regarding the fluid and radiation flows hold to high accuracy. The solutions in the different regions are matched at their interfaces, and the global solution is obtained once the boundary conditions are met at each time step. In what follows, we sketch the basis of this analysis and present some numerical results for the evolution of protostars. The actual computational details are somewhat involved and require a variety of perturbational techniques to effect a smooth and accurate transition from one region to another. These techniques are discussed in Papers II and III. Our procedure is the numerical analog of the analytical method of matched asymptotic expansions familiar to other branches of science (Van Dyke 1964; Cole 1968).

c) The Main Accretion Phase

In the published collapse calculations, a number of interesting phenomena are found to precede the main accretion phase. These include rarefaction waves (Westbrook and Tarter 1975), luminosity waves (Winkler and Newman 1980a, b), formation of a temporary core of molecular hydrogen (Larson 1969), further collapse and core bounce (Hayashi 1966), and formation of a stable hydrostatic core of partially ionized hydrogen. The detailed behavior of these processes depends strongly on the assumed initial conditions, and there is little hope of observing this detailed behavior in a real system. Therefore, as a first step toward simplifying the calculation, we will not compute this transient evolutionary stage. We shall assume that the collapse has already entered the main accretion phase in which a stable central core, taken to have an initial mass 0.01 $M_\odot$, accretes matter from a distended envelope.

Figure 1 shows schematically the structure of a protostellar cloud at an arbitrary but representative instant during the main accretion phase. Most of the cloud volume is contained in the outer envelope. This region exhibits interesting dynamical behavior; but its thermal response can be simply described. To a good approximation, as has been remarked upon by many authors (e.g., Larson 1973), the collapse in this optically thin region proceeds almost isothermally. This circumstance can be understood by considering the high temperature sensitivity of the grain emission rate. By slight variations in temperature, the grain cooling can offset either compressional heating or absorption of starlight over a wide range in density. During the main accretion phase, the dynamics of the outer envelope is dominated by the outward spreading of the falling mass toward the collapsed interior (Shu 1977; Hunter 1977). If a very luminous star appears at the center, reversal of the flow...
in the inner regions may occur through the action of radiation pressure on the dust grains (Yorke and Krugel 1977). For the low-mass protostars considered in this paper, reversal does not occur without the introduction of stellar winds.

The isothermal approximation begins to break down badly when densities become sufficiently high to trap the cooling infrared radiation. The corresponding radius is denoted the dust photosphere in Figure 1. The luminosity and effective temperature at this surface are the main quantities of interest for anyone attempting to observe an accreting protostar.

As the freely falling gas and dust cross the dust photosphere, this matter next enters the dust envelope, a region whose properties are largely set by the high opacity of the dust grains. In order to push out the luminosity generated in the interior, the temperature in the dust envelope must increase inward. This rising temperature destroys the grains at a radius \( r_d \), abruptly lowering the total opacity by a few orders of magnitude. From this dust destruction front to an inner radius \( r_t \), the gas is virtually transparent to the ambient radiation field (except at very large mass accretion rates). We call the space between \( r_d \) and \( r_t \) the opacity gap. The rising density and temperature of the gas as it flows inward through the gap may cause it to become optically thick again. If this happens before the radius \( R \) of the hydrostatic core is reached, the opacity gap will end at a gas photosphere whose radius \( r_d \) is greater than \( R \) (as shown in Fig. 1). The gas will then cross the optically thick radiative precursor and finally arrive at the accretion shock, where it is suddenly brought to rest before settling onto the hydrostatic core. If the gas does not become optically thick until after passing through the accretion shock, the gas photosphere is located at the shock front, \( r_d = R \). Behind the viscous shock, the postshock gas of very high temperature cools in a spatially thin radiative relaxation layer. We treat this layer via total jump conditions for the accretion shock (§ IIh). The relaxed gas is optically thick and generally has a specific entropy incompatible with the rest of the core. Thus, additional radiative adjustments usually have to be made diffusively in a highly subsonic settling zone before the newly added material is smoothly incorporated into the interior of the core.

This quick overview of protostellar structure during the main accretion phase provides an important lesson. The volume contained within the dust photosphere is a very small fraction of the entire cloud volume (typically one part in 10\(^5\)). Outside this radius, the infalling matter is optically thin and falling at supersonic speeds. Thus, this gas is unaffected, either radiatively or dynamically, by processes occurring within the dust photosphere. That tiny inner volume acts merely as a gravitational sink for the outer envelope. Conversely, the inner material is aware of the outer envelope solely
through the mass accretion rate $\dot{M}(t)$. The specification of this function, which replaces a detailed calculation of the outer envelope, is the departure point of our approach.

d) The Mass Accretion Rate

The overall magnitude of $\dot{M}$ has a very simple expression, while its detailed variation with time depends on a variety of factors, such as the nature of the outer boundary. For the isothermal collapse of a gaseous sphere whose initial state is not far removed from the critical state for gravitational stability, $\dot{M}$ has the order of magnitude

$$\dot{M} \sim a_T^{-3}/G, \quad (10)$$

where $a_T = (kT/m)^{1/2}$ is the isothermal sound speed of gas of mean molecular mass $m$.

A rough argument serves to justify equation (10). Any nonmagnetic nonrotating globule whose initial state corresponds approximately to thermal support against self-gravity will have comparable free-fall time scale and sound-crossing time:

$$t_f \sim \frac{R_c^{3/2}}{(GM_c^{1/2})} \sim \frac{R_c}{a_T} \sim t_{sc},$$

where $M_c$ and $R_c$ are the cloud mass and radius. The mass accretion rate $\dot{M}$ must have the order of magnitude,

$$\dot{M} \sim M_c/t_f,$$

from which we easily recover equation (10) after eliminating $R_c$.

More detailed analyses support this conclusion. The similarity solution for the gravitational collapse of a singular isothermal sphere gives a numerical coefficient in equation (10) of 0.975 (Shu 1977). For the collapse of more general isothermal spheres, a coefficient substantially greater than this value is attained early after core formation, but the mean value for the entire accretion phase is of order unity (Hunter 1977).

Equation (10) exhibits the seemingly surprising result that the mass accretion rate increases as the cube of the signal speed associated with the support of the initial (unstable) globule. That $\dot{M}$ should have this behavior merely means that the density at the onset of gravitational collapse must have been correspondingly larger in a cloud fragment which had higher initial hydrostatic support. From this discussion, we see that the value given by equation (10) represents the minimum value of $\dot{M}$ for the realistic problem. Any additional contribution to the hydrostatic support of a cloud—for example, from magnetic fields or turbulence—would increase the required density to initiate gravitational collapse, and thus would increase $\dot{M}$. We may heuristically account for these effects by an obvious generalization of equation (10):

$$\dot{M} \sim (a_T^{-2} + v_A^2 + v_t^2)^{3/2}/G, \quad (11)$$

where $v_A$ and $v_t$ are the Alfvén and turbulent speeds characteristic of the initial state. Numerical estimates of $\dot{M}$ based on equation (11) yield rates which range from $10^{-6}$ to $10^{-4} M_{\odot} \text{yr}^{-1}$ if we use conditions which are believed to prevail in molecular cloud fragments. Clearly, there exists a considerable range in values of $\dot{M}$ which could be considered to be physically plausible (see also Mouschovias 1977; Nakano 1980).

The temporal behavior of $\dot{M}(t)$ will show significant variation when the outer envelope begins to be depleted of material. Consider, for example, the case of the singular isothermal sphere. Here, $\dot{M}$ is an exact constant as long as the similarity solution holds. The similarity solution, however, breaks down when the "expansion wave," which separates falling matter from static matter, reaches the outer boundary (Shu 1977). This occurs when 0.488 of the original cloud mass has fallen into the core. If the influence of the outer boundary were ignored, the resulting mass could be plotted as a straight line as a function of time (curve 2 in Fig. 2). If, instead, we impose an outer boundary condition of constant pressure, we can expect the expansion wave to reflect into an inwardly traveling compression wave. Thus, the ambient pressure would help push the remaining material into the core and lead to higher late values of $\dot{M}$ (curve 1 of Fig. 2). If, on the other hand, we impose an outer boundary condition of constant volume, we can expect the expansion wave to reflect into an inwardly traveling rarefaction wave. Thus, the nonmoving gas near the boundary would

![Figure 2](image.png)

**Fig. 2.** The growth of the mass of the central hydrostatic core. Curve 2 illustrates a case of a constant mass accretion rate, such as that obtained for the collapse of a singular isothermal sphere when outer boundary conditions are ignored. Curve 1 gives a schematic increase in $\dot{M}$ in the late stages because of the application of a constant pressure at the outer boundary, while curve 3 gives a schematic decrease because of the requirement of a constant volume.
tend to retard the inflow and to lead to lower late values of \( M \) (curve 3 of Fig. 2). Although we have discussed only the singular isothermal sphere, complete numerical calculations can be expected to show this qualitative behavior for \( M(t) \) (e.g., see Winkler and Newman 1980a, b for a sample calculation with constant volume boundary condition).

Given the uncertainty concerning the detailed circumstances which trigger the formation of individual stars, we consider it wisest at present to adopt a policy which treats \( M(t) \) as a freely specifiable function. To begin, we shall use the simplest function possible, namely, an accretion rate \( \dot{M} \) which is constant in time. In a more complete calculation, the detailed specification of \( M(t) \) could, in principle, be supplied by a separate dynamical calculation of the outer envelope.

e) The Dust Photosphere

Let us consider more carefully the transition between the optically thin, nearly isothermal material in the outer envelope and the optically thick material further inside. At any time \( t \), the material at the base of the envelope is drawn inward by the gravitating mass at its center, eventually reaching free-fall speed onto a core of mass \( M_\star(t) \):

\[
M_\star(t) = \int_0^t \dot{M} dt. \tag{12}
\]

At a certain radius \( r \) in the inflow, the mean free path of an infrared photon near the peak of the local distribution will become comparable to \( r \). Outside this point, the energy carried by the outward traveling photons has little chance of diffusing back to the interior; consequently, we may define the radius of the (extended) dust photosphere by the condition:

\[
\rho \kappa_d r = 1, \tag{13}
\]

where \( \kappa_d \) is an appropriate mean opacity of dust at a matter temperature \( T \). In a static atmosphere, the pressure scale height \( H = kT/m_g \) would replace \( r \) in the above definition. In optically thin regions of an accretion flow, however, all thermodynamic scale heights are comparable with \( r \), and the condition (13) is more appropriate apart from an uncertainty about a factor of 2 because of the spherical geometry. Fortunately, low-temperature opacities are highly sensitive to temperature, and the temperature varies rapidly in the optically thick dust envelope. Thus, the transition radius from optically thick to optically thin conditions is reasonably well defined by equation (13) despite the numerical uncertainty of the right-hand side.

The dust photosphere is the outermost boundary of the region of interest to our evolutionary calculations.

We apply five conditions at this point:

\[
b = b_0, \quad u = -\left(\frac{2GM_\star}{r}\right)^{1/2}, \quad \rho = -\frac{\dot{M}}{4\pi r^2 u},
\]

\[
F_{\text{rad}} = \sigma T^4, \quad E_{\text{rad}} = aT^4 \text{ at } \rho kr = 1. \tag{14}
\]

The initial graphite grain radius \( b_0 \) can be taken to be that appropriate for bare graphite cores since the temperature at the dust photosphere turns out to be too high (> 100 K) to allow icy mantles. The free-fall condition of equation (14) holds for the inner limit of the solution in the optically thin outer envelope. In writing this relation, we have ignored the small amount of mass contained between the hydrostatic core and the dust photosphere. The conditions expressed by equation (14) are used to locate the radius \( r_\star \) of the dust photosphere and to provide starting values of \( b, u, \rho, T, \) and \( E_{\text{rad}} = 3P_{\text{rad}} \) for inward integrations of the inner envelope.

f) Steady State Approximation for the Accretion Flow

For a typical mass accretion rate of \( 10^{-5} M_\odot \text{ yr}^{-1} \), the luminosity produced at the surface of a core of roughly stellar mass and radius will be \( \sim 10^2 L_\odot \). Equation (14) then gives dust photospheres with radii \( r_\star \sim 10^{14} \text{ cm} \) and effective temperatures \( T_\star \) of a few hundred kelvins. It is then easy to calculate that it takes each fluid element about 1 yr to fall freely into the core. The photon diffusion time from the surface of the core to the dust photosphere turns out to be even shorter. Either time is much shorter than the 10^9 yr required for both the core and the outer envelope to evolve appreciably. Thus, to compute the behavior of the region between the dust photosphere and the core surface, we may consider \( \partial M/\partial t = \dot{M} \) in equation (1) to be independent of position \( r \), and we may neglect all other explicit time derivatives in equations (2)–(7). This approximation is an excellent one after the transient phase of core formation. Fortunately this phase is very short-lived; moreover, our final results can be shown to be asymptotically insensitive to the details of this transient stage (§ III).

The steady state approximation for both matter and radiation fields represents a considerable simplification of the computational problem. First, the reduction of the governing equations (1)–(9) to a set of ordinary differential equations allows the use of standard numerical integration schemes which easily provide the high accuracy needed to resolve spatial structures spanning many orders of magnitude of change in the relevant variables. Second, the accretion shock can now be treated as a mathematical discontinuity which satisfies certain jump conditions. Thus, we need not smear out the shock by means of artificial viscosity (cf.
Westbrook and Tarter 1975; Winkler and Newman 1980a, b). Artificial viscosity methods are generally ill-suited for problems involving strong radiating shocks because the proper resolution of the postshock radiative relaxation layer and the shock precursor region requires that these domains not be smoothed artificially on length scales of the order of a photon mean free path (§ IIa; see also Whitney and Skalakuris 1963; Zel'dovich and Raizer 1967).

**g) The Opacity Gap**

The existence of an opacity gap is an important feature of our solution. As the dust and gas flow inward, there comes a radius $r_d$ where equation (5) leads to a rapid and complete destruction of the graphite grains. The silicate grains generally will have disappeared even earlier; consequently, the total opacity drops by several orders of magnitude inside $r_d$. The gas is optically thin ($\rho_k r \ll 1$) between the dust destruction front, radiating at an effective temperature $T_d$, and an inner gas photosphere, radiating at an effective temperature $T_g$. Under these circumstances, the variable Eddington factor $f$ is approximately that appropriate to the inside of a cavity bounded by two spherical walls of temperatures $T_g$ and $T_d$ (Fig. 3):

$$f(r) = \frac{1}{3} \left( 1 - \frac{D \mu_g^3(r)}{1 - D \mu_d(r)} \right)$$

where

$$\mu_g(r) = \left( 1 - \frac{r^2}{r_g} \right)^{1/2}$$

and $D = \frac{T_d^4 - T_g^4}{T_g^4 + T_d^4}$. (15)

The hot dust destruction front should provide a “back warming” of the protostellar core. However, the effective temperature $T_g$ of the inner gas photosphere always exceeds $T_d$ (see eqs. [25] and [27]). In this situation, the back warming by the hot dust is a small effect to the extent that $T_d < T_g^4$. Moreover, an observer situated at the edge of the dust destruction front sees an equally hot “sky” in all directions except when facing the even hotter central protostellar core. Thus, the net flow of radiation is outward even at $r = r_d$ (Paper III; see also Milne 1930).

Although the opacity has a gap, the combination $(\rho_k r)(GM/\dot{R}r^2)$ turns out to be large enough to couple $aT^4$ in equation (3) fairly closely to the ambient radiation energy density $E_{\text{rad}}$ (Paper III). Since $E_{\text{rad}}$ rises at best as $r^{-2}$ in the opacity gap, the deceleration due to gas pressure will at best be proportional to $r^{-3/2}$ and cannot appreciably retard the inflow. In other words, the enthalpy flow of the gas will absorb little of the emergent radiative flux. The gas impacts the core at close to free-fall speed, and the shock-released luminosity will be given approximately by $GM_\ast M/R$ (cf. eq. [16c]). Moreover, because the preshock gas can be heated to internal energy levels which are only a small fraction of the free-fall energy, the surface pressure of the hydrostatic core must approximately be equal to the ram pressure of the freely falling matter, $(2GM_\ast/r)^{1/2} M/4\pi R^3$ (cf. eq. [16b]).

This discussion shows that the outer boundary of the protostellar core depends mostly on the global properties of the core plus the mass accretion rate $\dot{M}$. In this approximation, the evolution of the core is independent of the rest of the collapse calculation except for the single factor $\dot{M}$. The dust opacity enters only indirectly (although very importantly) by allowing a dust photosphere from which the accretion luminosity can ultimately escape without overly heating the infalling gas.

This circumstance provides a “rule of thumb” for the time-averaged radius $\tilde{R}$ of a protostar during the accretion phase. Suppose nuclear energy sources to play relatively little role during the time $t$ that it takes to build up the star to mass $M$ at an average accretion rate $\dot{M} = M/\tilde{t}$, and suppose that $M$ and $t$ are both low enough that a fully quasi-static model of the same mass $M$ and age $t$ would still be on a convective contraction track. Then the arguments of the Appendix suffice to show that $\tilde{R}$ must be less than 1.5 $R_H$, where $R_H$ is the radius of a Hayashi pre–main-sequence model of corresponding mass and age.

One of the calculations of § III serves to illustrate the rule of thumb. A protostar of mass $M = 1 M_\odot$ which
was accurately built up at a mass accretion rate $\dot{M} = 10^{-5} M_\odot \text{ yr}^{-1}$ has a final radius of 4.7 $R_\odot$, and a time-averaged radius of 3.6 $R_\odot$. A 1 $M_\odot$ pre-main-sequence model of age $10^5$ yr contracts, in contrast, to a radius of 4.4 $R_\odot$ (Iben 1965). This example suggests that age estimates of $T$ Tauri stars based on interpreting their H-R diagrams in terms of classical pre-main-sequence contraction tracks (Cohen and Kuhi 1979) are not likely to be too far wrong, provided we take such “ages” to be the time since stable core formation.

h) The Accretion Shock

In our formulation, the accretion shock is a discontinuity which separates the dynamic flow in the inner envelope from the hydrostatic central core. We wish to incorporate the radiative relaxation region, where the shocked gas cools back into thermal equilibrium with the radiation field, as part of the discontinuous transition. For this extended purpose, the usual Rankine-Hugoniot jump relations for viscous shocks are not sufficient. The total jump conditions require an additional consideration of the radiative transfer and cooling process.

Figure 4 shows schematically the profiles of gas and radiation temperatures in the vicinity of the accretion shock. As the gas flows hypersonically into the shock front, it is heated by the outflowing radiation to a temperature $T_1$ comparable to the ambient radiation temperature $T_{\text{rad}} \equiv (E_{\text{rad}}/a)^{1/4}$. At radius $R$, this gas, moving at velocity $u_1$, slams into the surface of a hydrostatic core of mass $M_s$. This gas passes through a spatially very thin viscous layer and is shocked to a high temperature $T_2$. If we ignore the slight amount of energy contained in the internal degrees of freedom, $T_2$ is given approximately by $(3/16)m_2u_1^2/k \approx (3/8)G M_s m_2 / R k$, where $m_2$ is the mean molecular mass at point 2. Thus, $T_2$ is comparable to the temperature at the center of the protostellar core, and must, therefore, relax to the much lower value appropriate for the surface of a star. It is important to note that the radiation temperature does not undergo a similarly large jump. Indeed, since the thickness of the viscous shock (about one particle mean free path) is much less than a photon mean free path, integration of the transfer equation across the viscous layer shows that the radiation field is closely preserved across the front. Thus,

$$[E_{\text{rad}}]^2 = [F_{\text{rad}}]^2 = [P_{\text{rad}}]^2 = 0.$$  

Since $E_{\text{rad}}$ will be much less than $aT^4$ downstream from the viscous shock, the vast difference between emission

\[1\] To be complete, we should note that our calculations are carried out with deuterium burning while Iben’s are not. This effect probably contributes on the 10–20% level to the final radius.

and absorption in equation (3) will cause the specific entropy of the gas to decrease quickly in the postshock flow. This occurs with about a 33% increase in the gas pressure as the gas decelerates to very subsonic values; therefore, cooling occurs roughly isobarically until $T$ approaches $T_2$ when the gas and radiation temperatures become equal in the dense regions which prevail below the surface of the protostar. Given this picture, we now want to derive the jump conditions which connect points 1 and 3.

We shall show below that the optical thickness of the radiative relaxation region is always much less than unity. In addition, the emissivity is usually large enough that the spatial thickness of the relaxation region is considerably less than the core radius $R$; thus, a plane-parallel analysis will suffice. We also assume that the flow is steady in the frame which moves with the velocity of the shock front, $v_1 = dR/dt$. With these approximations, the first three jump conditions express conservation of total mass, momentum, and energy:

\begin{align}
\rho_3(u_3 - v_c) &= \rho_3(u_1 - v_3), \tag{16a} \\
P_{\text{rad}}(3) + P_3 + \rho_3(u_3 - v_c)^2 &= P_{\text{rad}}(1) + P_1 + \rho_1(u_1 - v_3)^2, \tag{16b} \\
L_{\text{rad}}(3) - M_{\epsilon(3) + \frac{1}{2}(u_3 - v_3)^2} &= L_{\text{rad}}(1) - M_{\epsilon(1) + \frac{1}{2}(u_1 - v_3)^2}, \tag{16c}
\end{align}

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where $h$ is the specific enthalpy of the gas and $F_{\text{rad}} = 4\pi R^2 F_{\text{rad}}$ is the radiative luminosity in the shock frame. (Since $v_2$ is at most only a few cm s$^{-1}$, we shall henceforth not bother to distinguish this luminosity from that appropriate for our inertial frame.) Beside the three conditions (16), we derive two more jump conditions by first integrating equations (7) and (8) in the plane-parallel steady-state approximation:

$$F_{\text{rad}}(1) - F_{\text{rad}}(3) = \int_{v_2}^{v_3} \left( \frac{\kappa_{\text{em}}}{\kappa} aT^4 - \frac{\kappa_{\text{abs}}}{\kappa} E_{\text{rad}} \right) d\tau,$$

(17a)

$$P_{\text{rad}}(3) - P_{\text{rad}}(1) = \int_{v_2}^{v_3} \frac{1}{c} F_{\text{rad}} d\tau,$$

(17b)

where $d\tau = -\rho d\tau$ with $\kappa = \kappa_{\text{em}} + \kappa_{\text{abs}}$. To obtain equation (17a), we have noted that the work done by the radiative force is negligible compared to the energy imbalance between emission and absorption. We have also made use of the conservation of the radiation field across the viscous shock 1→2. At point 3, we now impose the requirement that the radiation field be isotropic and thermally well coupled to the matter:

$$E_{\text{rad}}(3) = 3P_{\text{rad}}(3) = aT_3^4.$$

(18)

Consider now the implications of equation (17a). Since $F_{\text{rad}} \leq cE_{\text{rad}}$ and $aT^4 \gg E_{\text{rad}}$ throughout most of the radiative relaxation layer, equation (17a) requires

$$\tau_3 - \tau_2 \ll 1.$$

Thus, as was claimed earlier, the emissive relaxation of the matter temperature $T$ to $T_{\text{rad}}$ must occur, for a strong shock, within a space which occupies much less than one photon mean free path. Equation (17b) now implies that the difference between $P_{\text{rad}}(3)$ and $P_{\text{rad}}(1)$ cannot exceed $(\tau_3 - \tau_2) F_{\text{rad}}(1)/c$, which, in turn, must be much less than either term individually. (See eqs. [18] and [24] for the worst case of optically thin preshock conditions.) Thus, to a high degree of approximation,

$$P_{\text{rad}}(1) = P_{\text{rad}}(3).$$

(19)

Equation (19), together with equation (16b), clearly expresses the conclusion that across a span which is optically very thin, the momenta contained in the matter and radiation fields are separately conserved. Equation (19) provides one of the two required radiation jump conditions.

The other radiation jump condition takes a form which depends on whether the preshock point 1 is thin or thick to optical photons (i.e., whether $\rho_1 \kappa_1 R \ll 1$ or $\rho_1 \kappa_1 R \gg 1$, where $\kappa_1$ is evaluated as a Rosseland mean at the density and temperature of the gas at the preshock conditions). During the main accretion phase,

**Fig. 5.—Radiative jump conditions for the accretion shock.** (a) In the case of optically thin preshock conditions, $\rho_1 \kappa_1 R \ll 1$, an outward component of the flux $\sigma T_g^4 + 3(F_t - F_3)/4 + F_3$ is opposed by the inward contributions: $\sigma T_g^4$ in near-infrared photons from the dust destruction front, $(F_t - F_3)/4$ in optical photons from X-rays reprocessed in front of the shock, and $(F_t - F_3)/2$ in shock-deposited soft X-rays. The outward optical flux $\sigma T_g^4 + 3F_t/4 + F_3/4$ from behind the shock plus the outward optical contribution $(F_t - F_3)/4$ from soft X-rays reprocessed in front of the shock yields the optical flux $\sigma T_g^4 = F_3 + \sigma T_d^4$ from the gas photosphere seen by an observer in the opacity gap. (b) In the case of optically thick preshock conditions, $\rho_1 \kappa_1 R \gg 1$, two optically thick regions—the radiative precursor and the postrelaxation layer—are connected by an optically thin span, the radiative relaxation layer. Thus, the temperatures $T_1$ and $T_3$ must be equal: $T_3 = T_1$. In this case, the gas photosphere must occur roughly at $\rho_1 \kappa_1 R = 1$ in front of the radiative precursor, where the effective temperature $T_g$ is given by $\sigma T_g^4 = F_{\text{rad}} + \sigma T_d^4$. 

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the preshock layers are invariably thick to the soft X-ray photons generated inside the relaxation layer. Thus, the X-ray photons leaving the radiative relaxation layer will be reprocessed into optical photons appropriate for the gas temperature of the surrounding layers. What is the postrelaxation temperature $T_2$ under these conditions? (Consult Fig. 5.)

To answer this question, let us first consider the case when point 1 is thin to optical photons. If the preshock layers are optically thin, outwardly traveling optical photons are free to fly past radius 1. Inwardly traveling near-infrared photons from the dust destruction front or optical photons from reprocessed soft X-rays will also be free to fly inwards to radius 3. For values of $(\tau - \tau_s)$ which are not too large, we may perform an outward integration from the subphotospheric layers to point 3, ignoring the difference between $F_{\text{rad}}$ and $F_{\text{rad}}$ (of order $d[\tau - \tau_s]/(1/c)F_{\text{rad}}$). Because of the continuous settling of material, however, we cannot a priori assume radiative equilibrium. For steady settling at very subsonic speeds, the luminosity decreases inward into the star as $L_{\text{rad}} = L_3 - M((\Phi_3 - \Phi) + (h_3 - h))$, where $\Phi$ is the gravitational potential. Fortunately, as long as $L_3 \ll L_1$, the functional form of the interior luminosity will turn out to have little influence on the determination of the boundary temperature $T_3$. Thus, for simplicity of discussion, we shall assume $F_{\text{rad}} = \text{constant} = F_3$ for the range of $\tau$ of interest. Then, we can integrate equation (8) under the radiation condensation approximation to obtain the usual linear relation between $T^4$ and $\tau$:

$$aT^4 = \frac{3F_3}{c} (\tau - \tau_s) + E_3,$$  

(20)

where we have written $F_3$ and $E_3$ for $F_{\text{rad}}(3)$ and $E_{\text{rad}}(3)$. With a source function given by $c/4\pi$ times equation (20), the outwardly directed specific intensity at point 3 can be obtained as

$$I(\tau_3, \mu) = \frac{3F_3}{4\pi} \mu + \frac{cE_3}{4\pi}$$ for $\mu > 0$.  

(21)

We now evaluate the constant $E_3$ by the requirement that the first angular moment of $I(\tau_3, \mu)$ should yield the net flux $F_3$:

$$F_3 = 2\pi \int_0^1 I(\tau_3, \mu) \mu d\mu + 2\pi \int_{-1}^0 I(\tau_3, \mu) \mu d\mu.$$ 

(22)

The first integral in equation (22) can easily be evaluated to be $F_3/2 + cE_3/4$ by substituting equation (21). To compute the second integral requires us to count all the incoming photons. The total rate of energy release per unit area by shock deposition is $F_1 - F_3$, as obtained from equation (16c) upon division by $4\pi R^2$. Half of the X-rays generated between $\tau_1$ and $\tau_2$ flow outward; half, inward. The $(F_1 - F_3)/2$ in soft X-rays which fly outward will be absorbed by the preshock gas within about $10^7$ cm, and this X-radiation will be reprocessed into optical photons (Fig. 5a). If we assume the reradiation to be isotropic and the preshock layers to be thin with respect to optical photons, half of the reprocessed photons will fly back toward the star. Thus, the total contribution to the second integral in equation (22) will consist of $-(F_1 - F_3)/2$ in soft X-rays from the radiative relaxation layer, $-(F_1 - F_3)/4$ in reprocessed optical photons from the preshock layers, and $-\sigma T_d^4$ in near-infrared photons from the dust destruction front. Equation (22) therefore becomes

$$F_3 = \left[ \frac{1}{2}F_3 + \frac{1}{4}cE_3 \right] - \left[ \frac{1}{4}(F_1 - F_3) + aT_d^4 \right],$$ 

(23)

which we may solve for $cE_3/4 = \sigma T_d^4$, with $a = c a/4$, to obtain

$$T_3 = \left[ \frac{1}{a} \left( \frac{3}{4}F_1 - \frac{1}{4}F_3 + T_d^4 \right) \right]^{1/4} \text{if } \rho_1 \kappa_1 R \ll 1.$$ 

(24)

Notice that if $F_1 = F_3$ (no shock-deposited luminosity) and if $\sigma T_d^4 = 0$ (no hot sky), equation (24) would yield the standard formula for the boundary temperature of a normal stellar atmosphere (in the Eddington approximation). In the actual situation relevant to low-mass protostars, $F_1 \gg F_3$, and the interior luminosity plays little role in determining the boundary temperature $T_3$. The accretion shock serves to raise $T^4$ at every point in the subphotospheric layers by a constant amount in accordance with equation (20). These layers then radiate an outward flux of optical photons equal to

$$2\pi \int_0^1 I(\tau_3, \mu) \mu d\mu = \frac{1}{12}F_3 + \frac{1}{4}cE_3 = \frac{1}{2}F_1 + \frac{1}{4}F_3 + \sigma T_d^4.$$ 

The opacity gap sees an additional optical contribution from X-ray reprocessing in a thin layer in front of the shock by an amount $(F_1 - F_3)/4$. Thus, the total outward flux to be used in the expression (15) for the variable Eddington factor $f$ is given by the sum of these two contributions, i.e.,

$$\sigma T_g^4 = F_1 + \sigma T_d^4.$$ 

(25)

The net outward flux after subtracting the inward contribution $\sigma T_g^4$ of near-infrared photons from the dust destruction front is, of course, $\sigma(T_g^4 - T_d^4) = F_1$. The isothermal tendency of the resulting gas "photosphere" explains our neglect of the effects of limb darkening in deriving equation (15). The isothermal tendency would be reinforced for the subphotospheric layers if the internal radiative flux $F_{\text{rad}}$ decreases with increasing $\tau$ (Anderson and Shu 1978), as would be the case with a settling flow (Paper II).

In our early numerical calculations, we used a heuristic argument to obtain equation (25), but we mistakenly adopted an approximation that essentially
equated the postrelaxation temperature $T_b$ to the effective temperature $T_0$. A belated explicit recognition that soft X-ray photons must be present in the spectrum of the outwardly directed radiative point at point 1 (see also Ulrich 1978) led us to the derivation given above. The numerical difference between $T_0 \approx (F_1/a)^{1/4}$ and $T_0 \approx (3F_1/4a)^{1/4}$ is only about 7%; consequently, our use of the earlier incorrect boundary temperature had little practical impact on the final results. Zel'dovich and Raizer (1967) found an expression similar to equation (24) in their discussion of "subcritical" shocks. Their problem concerned relatively weak shocks (under optically thick preshock conditions), which makes their derivation inapplicable to protostars. Larson (1969, Appendix B) used a heuristic argument to derive jump conditions for the protostellar problem. However, his former derivation suffered from a number of unnecessary assumptions, the most suspect being the claim that the ratio $cP_{rad}/F_{rad}$ always exceeds $\frac{1}{4}$. In fact, although $cE_{rad}/F_{rad}$ cannot be less than unity, no similar limit exists a priori for $cP_{rad}/F_{rad}$. In any case, Larson's jump condition did not include the effects of "back warming" in a quantitatively precise manner, but his numerical results would have been little influenced by these corrections.

Consider now the opposite limit when the preshock conditions are thick to optical photons. If the preshock point satisfies $\rho_1 \kappa R \gg 1$, we expect the ambient radiation field to be isotropic ($f = \frac{1}{2}$) and also to be well coupled thermally to the matter ($T_{rad} = T$). Under these conditions, $P_{rad}(1) = aT_1^{4}/3$, and equations (18) and (19) imply

$$T_3 = T_1 \quad \text{if} \quad \rho_1 \kappa_1 R \gg 1.$$  \hspace{1cm} (26)

This provides the second radiation jump condition for the case $\rho_1 \kappa_1 R \gg 1$. The physical interpretation of equation (26) is given in Figure 5b.

Equation (26) verifies the intuitive result that two optically thick regions (the precursor and the postrelaxation layers) separated by much less than one photon mean free path must have identical temperatures. In this case, outward integrations from the core may continue across points 1–3 as if the shock had occurred isothermally (see Zel'dovich and Raizer's discussion of "supercritical" shocks). Only upon reaching the head of the radiative precursor would one expect the photospheric condition:

$$F_{rad} = \sigma(T_e^4 - T_d^4) \quad \text{at} \quad \rho_0 \kappa_1 = 1.$$  \hspace{1cm} (27)

In our calculations, which extend to the dust photosphere, equation (27) serves to define the radius $r_e$ and effective temperature $T_e$ that enter in equation (15) when a radiative precursor is present. The gas temperature $T$ at $\rho_0 \kappa_1 = 1$ in such a dynamic atmosphere need not exactly equal $T_e$.

We have undertaken a lengthy discussion of the radiative dynamics of accretion shocks because it is a source of considerable controversy (see, e.g., Hayashi 1970). However, our analysis shows that the practical difference between the jump conditions (25) and (26) is not of a decisive character for the problem of protostellar evolution. The most important point in this regard is the existence of an opacity gap. Compared to this effect, the issue of whether the preshock conditions are optically thin or optically thick fades into relative unimportance. The latter issue merely determines whether the photons generated in the shock layer are free to fly immediately away or whether they first have to diffuse through an optically thick radiative precursor to the gas photosphere where they can then freely escape.

\begin{flushright}
\textit{i) The Hydrostatic Core}
\end{flushright}

After the infalling gas crosses the radiative relaxation layer of the accretion shock, its velocity becomes extremely subsonic. If we use $M(r, t)$ as the Lagrangian mass coordinate, we may identify the substantial derivative as

$$\frac{\partial}{\partial t} + u \frac{\partial}{\partial r} = \left( \frac{\partial}{\partial t} \right)_M.' \hspace{1cm} (28)$$

If we now drop all other terms dependent on $u$ and adopt the approximation of good thermal coupling between matter and radiation, equations (1)–(4) and (6)–(9) become the usual equations of stellar structure. In particular, the sum of the time-dependent heat equations of the matter and radiation fields yields the total heat equation:

$$T \left( \frac{\partial s_{tot}}{\partial t} \right)_M = - \left( \frac{\partial L_{tot}'}{\partial M} \right)_t + \epsilon, \hspace{1cm} (29)$$

where $s_{tot} = s_{rad} + s$ is the total specific entropy and $L_{tot}' = L_{rad}' + L_{conv}' + L_{cond}'$ is the sum of the diffusive luminosities of the combined system.

The calculation of $(\partial s_{tot}/\partial t)_M$ can, in principle, be carried out in the standard manner of stellar evolution theory. In practice, there are two complications. First, for a low-mass protostar, the shock-deposited luminosity is much greater than the internal luminosity $L_3$ emerging from point 3. Nevertheless, the latter cannot be ignored, for it sets the value of the specific entropy of the newly added material after it slowly settles into the interior of the protostar. Until deuterium burns, the deep interior of the protostar is very nearly adiabatic. This set of circumstances produces extremely sharp spikes in the entropy and luminosity structure of the settling zone; the proper resolution of these spikes and the quasi-adiabatic regions which follow them require nonstandard techniques for treating the thermal problem.
Second, a protostellar core has the peculiarity of a variable mass. In a finite difference scheme using a time interval $\Delta t$, the outer layer of mass $M \Delta t$ did not exist as part of the core at the previous time step. There arises, therefore, a real difficulty in computing the term $(\delta s_{\text{rad}}/\delta t)_M$ for this “new mass.” This difficulty presumably explains why most calculations involving mass accretion arbitrarily adopt either the assumption that the specific entropy of the newly added material is identical to that just beneath it (e.g., Flannery and Ulrich 1977; Kippenhahn and Meyer-Hofmeister 1977) or the assumption that radiative equilibrium prevails down to some substantial mass zone (e.g., Benson 1970; Neo et al. 1977). The existence of entropy and luminosity spikes reveals both these assumptions to be invalid for protostar evolution. In Paper II, we shall discuss our techniques for dealing with these issues.

III. SUMMARY OF RESULTS

a) Synopsis of Computational Procedure

With given mass accretion rate $\dot{M}$, our integrations proceed essentially as follows:

1) At each time step $\Delta t$, we guess values for the postrelaxation pressure $P_r$ and temperature $T_r$.

2) These two surface values plus the inner boundary conditions, $r=0$ and $L'=0$ at $M=0$, and the core model of the previous time step allow us to solve the four stellar structure equations for the new core. The mass of the new core is $M \Delta t$ greater than its previous value.

3) We difference new and old core radii to obtain the shock velocity $u_s$. The postrelaxation fluid velocity $u_s$ can now be obtained from knowing $\rho_s$ and $R$, since $M$ represents the mass flow rate across the shock front:

$$\rho_s(u_s - v_s)4\pi R^2 = -\dot{M}. \quad (30)$$

4) We are now in a position to jump the shock to get upstream values at point 1 (eqs. [16]–[19] plus eq. [24] or [26]). The choice between equation (24) and (26) depends on whether the preshock material satisfies $\rho_1 k_1 R < 1$ or $\rho_1 k_1 R > 1$. If the former, we take $T_s$ in equation (24) to have its value at the previous time step.

5) With the values of all the quantities known at point 1, we may calculate the net energy outflow at this point:

$$L_e = L_{\text{rad}} - \dot{M}(h + \frac{1}{2}u^2 - GM_*/r). \quad (31)$$

The steady-state approximation for the inner envelope allows us to take $L_e$ to be a constant out to the dust photosphere.

6) The definition $\rho_k r T = 1$ plus equation (31) and the five conditions of equation (14) suffice to yield the values of $r$, $F_{\text{rad}}$, $b$, $u$, $\rho$, $T$, and $E_{\text{rad}}$ at the dust photosphere. We use these values (together with $P_{\text{rad}} = E_{\text{rad}}/3$ in optically thick regions) to start an inward integration through the dust envelope.

7) When equation (5) causes the graphite grain radius $b$ to reach zero, we have arrived at the dust destruction front. The matter temperature at this point is defined to be $T_\gamma$ (to be used in the next time step).

8) We continue the inward integration through the opacity gap, using $f$ as computed by equation (15) for the last time step. We switch to optically thick equations appropriate for the radiative precursor if $\rho_k r T$ should become larger than unity. If self-consistency is achieved (see the discussion below), the effective temperature $T_e$ defined at $\rho_k r T = 1$ by equation (27) will approximately equal both the matter and the radiation temperature.

9) Eventually, we arrive at $R$, point 1. We compare the values of $\rho$, $u$, $T$, and $L_{\text{rad}}$ obtained by the inward integration with those obtained in step 4. If any two of them agree, the other two will also agree because of the two explicitly conserved quantities, $L_e$ and

$$M = \frac{\rho \pi r^4}{2}. \quad (32)$$

10) In general, any two of the quantities tested for agreement will not, in fact, match the values obtained in step 4. When this happens, we go back to step 1, adjusting $P_r$ and $T_r$ by a Newton-Raphson procedure until agreement is found. Usually slight adjustments around the values

$$P_r \approx \frac{\dot{M}}{4\pi R^2} \left( \frac{2GM_*}{R} \right)^{1/2}, \quad 4\pi R^2 \sigma T_3^4 \approx \frac{3GM_* \dot{M}}{4R} + \frac{L_3}{2}. \quad (32)$$

suffice to make the core and inner envelope calculations self-consistent. We are then ready to take another time step.

Once the loop (1)–(10) is completed for any given time step, the quantities $E_{\text{rad}}$ and $P_{\text{rad}}$ will receive their correct values throughout the envelope even though they were never explicitly matched to the interior solutions. This statement is obviously true in optically thick regions, where $E_{\text{rad}} = 3P_{\text{rad}}/4T^4$ guarantees the correct values for $E_{\text{rad}}$ and $P_{\text{rad}}$ if $T$ is correct. The statement is also very nearly true in the optically thin opacity gap, where the variable Eddington factor was chosen in equation (15) to give exact expressions for $E_{\text{rad}}$ and $P_{\text{rad}}$ in the limit of vanishing opacity in the gap. For finite

2We should note a slight inconsistency in our treatment. In our inner envelope calculations, we define $M$ to be the mass which crosses a fixed radius per unit time; therefore at point 1, $M = -4\pi R^2 \rho u$. For our core calculations, we take $M$ to be the time rate of change of mass of the core, i.e., $M = -4\pi R^2 \rho(u - v_s)$. A slight error arises if the shock radius is not fixed in time, i.e., if $v_s = 0$. We have not bothered to correct for this effect because $v_s$ is, at most, a few cm s$^{-1}$ while $u_1$ is typically a few $\times 10^7$ cm s$^{-1}$. 

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opacity, $f$ will be slightly incorrect, but the error in $E_{\text{rad}}$ and $P_{\text{rad}}$, obtained by integrating equation (8), will be even smaller. Thus, in our actual calculations, we have not bothered to iterate (via the equation of transfer) to find improved Eddington factors.

Before we proceed to discuss some numerical examples, we wish to reiterate a previous point. To the extent that equations (32) can be used to replace the more precise exterior calculations, the core and envelope integrations are decoupled. In our actual calculations, the approximate relations (32) are used only as initial guesses for the full iteration procedure, but such precision can be abandoned without much practical sacrifice.

b) Examples of Low-Mass Protostar Models

We shall present detailed results of the structure of protostars during the main accretion phase in Papers II and III. Here, we wish only to summarize the global picture. Figure 6 shows a spacetime diagram of central events in a protostar calculation carried out with a mass accretion rate $M = 10^{-7} M_\odot$. The zero of time corresponds to the instant of core formation. The calculation begins at a time $t = 10^{-3}$ yr when the core mass is $0.01 M_\odot$. The solid curves pertain to an initial state (to be described in detail in Paper II) where the radius of the $0.01 M_\odot$ core was taken to be $3.45 R_\odot$.

The time development of the core radius as the core accretes more mass can be obtained by following the locus of the shock front. The core radius increases slightly at first, then decreases for a while, and finally follows a slow and steady increase in time. At the end of the calculation, when the core mass is $1 M_\odot$ ($t = 10^5$ yr), the core radius has become $4.72 R_\odot$.

The dashed curve gives the corresponding location of the shock front for an initial core which has a radius of $4.44 R_\odot$. Notice that the responses of the two different cores to the same mass accretion rate lead to states which converge asymptotically in time. In particular, despite the initial increase in core radius of the solid curve and the initial decrease of the dashed curve, by $t = 10 \times 10^3$ yr (when the core mass is $0.1 M_\odot$), the two cores have nearly indistinguishable interiors (except for the innermost parts). This circumstance leads us to conclude that the final outcome of protostar evolution is insensitive to the details of the transients which precede core formation as long as the total mass accreted is an order of magnitude or more greater than the initial mass of the core.

Because of this asymptotic convergence in time, Figure 6 plots additional events only for the case of the solid curves. Thus, except for minor changes, the dust photosphere and dust destruction front remain almost fixed in space. The increasing mass of the core leads, however, to an ever increasing accretion luminosity, and the latter heats the matter passing through the dust photosphere and the dust destruction front to ever higher temperatures. At $t = 10^3$ yr, the temperatures at the dust photosphere and the dust destruction front are, respectively, $116$ K and $1710$ K; by $t = 10^5$ yr these temperatures have become $414$ K and $2310$ K.

The core also exhibits interesting structure. The initial core of $0.01 M_\odot$ was constructed to have an entropy gradient $(\partial s_{\text{rad}}/\partial M) > 0$ which is stable to convection. This distribution of specific entropy turns out to be preserved throughout the main accretion phase because the small radiative and conductive fluxes have associated thermal time scales which are much longer than the duration of our calculation interval ($10^5$ yr). Thus, as more and more mass is added onto the core, the central part acquires a nearly spatially uniform pressure distribution, which, coupled with a specific entropy distribution that increases outward, leads to a temperature inversion. Such a temperature inversion was also found in the calculations of Winkler and Newman (1980 a, b). In any case, this temperature inversion in the central (partially degenerate) regions with its consequent implication of a temperature maximum at $M = 0$ leads to an off-center ignition of deuterium at $t = 20 \times 10^3$ yr. The release of heat raises the specific entropy of the ambient layers and, at $t = 36 \times 10^3$ yr, leads to the birth and outward spreading of an inner convection zone.

Outer convection zones also appear in our calculation. These occur whenever the gas which passes through the radiative accretion shock and the settling zone has a lower specific entropy than that present in the deeper layers. If we use the initial model of $3.45$

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The above discussion emphasizes the importance of being able to follow the entropy history of the matter in the settling zone, an issue which we shall take up again in Paper II. For now, we remark that the high-temperature gas photosphere that allows the escape of the shock-deposited luminosity $L_1 - L_2$ acts perversely as a bottleneck for the convective contribution to the interior luminosity $L_3$. Thus, we find the interior luminosity during the main accretion phase to be typically a factor of 10 or more below what it would have been if the accretion shock had not been present.

Figure 7 shows the history of our protostar’s evolution in the H–R diagram. During the main accretion phase, there are two photospheres: an observable dust photosphere at a low effective temperature $T_e$ (solid curve), and an unobservable gas photosphere at a high effective temperature $T_e$ (dashed curve). The dots on the dust photosphere track label the times 1, 2, ..., $10 \times 10^4$ yr when the central core has mass 0.1, 0.2, ..., 1.0 $M_\odot$. The corresponding points on the gas photosphere are higher by about 0.1–0.4 $L_\odot$. The decrease in radiative luminosity outward arises from the energy required for the dissociation of molecular hydrogen and from the work done by the radiation field as the gas and dust flow inward.

The gas photosphere does not become observable until the main accretion phase has ended. The way in which the central core first becomes visible as an optical object depends, therefore, on the detailed

![H-R diagram](image-url)
manner in which the accretion flow is cut off (or, perhaps, reversed by the onset of a stellar wind). Lacking a preferable procedure, we have examined the simplest case where the accretion flow is shut off abruptly when the core mass equals 1 $M_\odot$. The details will again be given in Paper II. Here, let us merely comment that the gas photosphere occupies, at that instant, the top of the dashed track and corresponds to a radius of 4.72 $R_\odot$, an effective temperature of 7310 K, and a (shock plus interior) luminosity of 66.0 $L_\odot$. In less than 1 day after the accretion is shut off, the gas photosphere cools along a locus of constant radius to an effective temperature of 4250 K and a luminosity of 6.39 $L_\odot$. After sitting at this lower point for another 3000 yr, the small amount of deuterium remaining in the convection zone is burned, and the star begins to descend what is essentially a conventional pre-main-sequence contraction track (Iben 1965).

Our decision to terminate the accretion at 1 $M_\odot$ was completely arbitrary. We could have terminated the accretion later, or earlier. This flexibility is another advantage of our technique. On the way to building up a 1 $M_\odot$ star, we have also built up, of course, stars of masses 0.1 $M_\odot$, 0.2 $M_\odot$, etc. If we had shut off accretion abruptly at these lower masses, we would have obtained a similar scenario to the 1 $M_\odot$ case. The approach to pre-main-sequence tracks would differ because stars of less than 0.5 $M_\odot$ would not yet have become essentially completely convective, stars of less than 0.3 $M_\odot$ would not yet have initiated deuterium burning, etc. But these differences can be handled by standard stellar evolution techniques. Perhaps phenomena like FU Orionis (Herbig 1966) would arise naturally from studies of the later phases of evolution of such objects. The important point for us here is that a single calculation by our technique exhausts all protostar evolutionary histories (for masses less than the final mass) with that mass accretion rate. In particular, a cursory check of our Figure 7, subject to the above comments, leads to the conclusion that our post-accretion tracks are in reasonable agreement with observations of T Tauri stars (Cohen and Kuhi 1979). To populate the H-R diagram with optically visible stars as high as some of those seen in the Taurus-Auriga complex would probably require us to adopt values of $M$ closer to $10^{-4} M_\odot$ yr$^{-1}$ (see Appendix), but such accretion rates would not be incompatible with what we know about molecular clouds (§ II'd).

IV. DISCUSSION

With respect to the controversy reviewed in § I, our investigation of the evolution of protostars obtains stellar radii at the end of the main accretion phase which are closer to the small values advocated by Larson (1969, 1972), Appenzeller and Tscharnuter (1975), and Winkler and Newman (1980a, b) than the very large values obtained by Narita, Nakano, and Hayashi (1970) and Westbrook and Tarter (1975). Nevertheless, our study does not represent a break with Hayashi's line of work as much as a continuation of it. In particular, our discussion of the opacity gap (§ IIg and Appendix) reestablishes the central importance for protostellar evolution of Hayashi's insight concerning the existence of a minimum temperature for gas photospheres. This explains why our models approach the pre-main-sequence tracks in the H-R diagram from the left (see Fig. 7). We note that Hayashi's "forbidden zone" should be supplemented by the recognition that dust photospheres have a maximum temperature governed by grain destruction. The "forbidden strip" between Hayashi's rightmost boundary for gas photospheres and our leftmost boundary for dust photospheres represents what we have called the opacity gap (see Fig. 6).

The theoretical work most directly comparable to ours is that carried out independently by Winkler and Newman (1980a, b). Apart from their not having included the effects of deuterium burning and convection, the results of Winkler and Newman substantially agree with our own. The difference in core radii obtained by the two groups arises primarily because of the difference in the effective accretion rates. According to the analysis of the Appendix, the core radius after the main accretion phase should scale roughly as $M^{1/3}$. We should note in this context that the larger final core radii of our models are in better agreement with the latest observations of T Tauri stars (Cohen and Kuhi 1979).

We feel, however, that our most important contribution is not so much the numerical resolution of the existing controversy in protostellar evolution, as the introduction of a series of semianalytic techniques that reduce the spherical accretion problem to a calculation which is no more difficult, in principle, than that encountered in ordinary stellar evolution theory. Techniques very different from the standard ones are required to solve the resulting set of differential equations and interface conditions, and these will be discussed in Papers II and III. We hope that this paper has already provided a glimpse of the gain in physical insight possible with our method.

We are grateful for informative conversations with P. Bodenheimer, M. Cohen, J. Gaustad, D. Hollenbach, L. Kuhi, and P. Woodward on problems of star formation. The numerical calculations for this work were performed at the Berkeley Computing Center. This research is supported in part by the National Science Foundation.
EVOLUTION OF PROTOSTARS

APPENDIX

APPROXIMATE FORMULA FOR RADIUS OF PROTOSTELLAR CORE

An approximate calculation for the time-averaged radius of the core during the accretion phase can be obtained by requiring the total energy radiated by the gas photosphere to equal the accretion energy:

\[ \int_0^t 4\pi R^2 \sigma T_g^4 \, dt' \approx \int_0^t \frac{GM_* M}{R} \, dt'. \]  
(A1)

For the above approximation to be valid, there must be an appreciable opacity gap with the gas photosphere at or close to the radius \( R \) of the protostellar core. Moreover, the enthalpy of the gas, including dissociation and partial ionization, must be small compared to the kinetic energy of free fall. For a constant mass accretion rate \( M \), the instantaneous core mass is given by \( M(t') = Mt' \). Equation (A1) now yields

\[ \bar{R} \approx \left( \frac{GM^2}{8\pi\sigma\bar{T}_g^4 t} \right)^{1/3}, \]  
(A2)

where \( \bar{R} \) and \( \bar{T} \) are appropriately time-averaged values of the radius and effective temperature of the gas photosphere, and the final mass of the core is \( M = Mt \).

Let us compare the result (A2) with the radius \( R_H \) of a quasi-statically contracting model of the same mass \( M \) and age \( t \). Assume this pre–main-sequence model to be descending a completely convective track with a virtually constant effective temperature \( T_H \) given by the boundary of Hayashi’s forbidden region in the H-R diagram. In accordance with the virial theorem, half of the gravitational potential energy of this model must be radiated away during the contraction. Thus, if we approximate the instantaneous structure to be a polytrope of index 1.5, we require

\[ 4\pi R_H^2 \sigma T_H^4 \approx \frac{d}{dt} \left( \frac{3}{7} \frac{GM^2}{R_H} \right) \]  
(A3)

Equation (A3) may be integrated with \( T_H = \) constant to yield the solution

\[ R_H \approx \left( \frac{GM^2}{28\pi\sigma T_H^4 t} \right)^{1/3}, \]  
(A4)

where we have formally taken \( R_H \) to be infinite at \( t = 0 \). Division of equation (A2) by equation (A4) now gives

\[ \frac{\bar{R}}{R_H} \approx \left( \frac{7 T_H^4}{2 \bar{T}_g^4} \right)^{1/3}. \]  
(A5)

Since \( T_H \) essentially represents the minimum possible effective temperature for any gas photosphere, \( \bar{T}_g \) must be larger than \( T_H \). Thus, formula (A5) implies that \( \bar{R} \) must be less than 1.5 \( R_H \), which is the “rule of thumb” quoted in § IIg. To check equation (A2) numerically, notice that \( M = 1 \, M_\odot, t = 10^3 \, \text{yr}, \) and \( \bar{T}_g = 7000 \, \text{K} \) yields \( \bar{R} = 4 \, R_\odot \) from equation (A2), a value which agrees well with the time-averaged radius of the protostar (§ IIg). To be fair, however, we should note that without a detailed calculation, we would not know that \( \bar{T}_g = 7000 \, \text{K} \) is a more reasonable value than, say \( \bar{T}_g = 10,000 \, \text{K} \).

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