Nonlocal conductivity in type-II superconductors

Chung-Yu Mou
Department of Physics, University of Virginia, McCormick Road, Charlottesville, Virginia 22901

Rachel Wortis
Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801
and AT&T Bell Laboratories, Murray Hill, New Jersey 07974

Alan T. Dorsey
Department of Physics, University of Virginia, McCormick Road, Charlottesville, Virginia 22901

David A. Huse
AT&T Bell Laboratories, Murray Hill, New Jersey 07974
(Received 5 October 1994)

Multiterminal transport measurements on YBa$_2$Cu$_3$O$_7$ crystals in the vortex-liquid regime have shown nonlocal conductivity on length scales up to 50 microns. Motivated by these results we explore the wave vector (k) dependence of the dc conductivity tensor, $\sigma_{\mu\nu}(k)$, in the Meissner, vortex lattice, and disordered phases of a type-II superconductor. Our results are based on time-dependent Ginzburg-Landau (TDGL) theory and on phenomenological arguments. We find four qualitatively different types of behavior. First, in the Meissner phase, the conductivity is infinite at $k = 0$ and is a continuous function of $k$, monotonically decreasing with increasing $k$. Second, in the vortex-lattice phase, in the absence of pinning, the conductivity is finite (due to flux flow) at $k = 0$; it is discontinuous there and remains qualitatively like the Meissner phase for $k > 0$. Third, in the vortex liquid regime in a magnetic field and at low temperature, the conductivity is finite, smooth and nonmonotonic, first increasing with $k$ at small $k$ and then decreasing at larger $k$. This third behavior is expected to apply at temperatures just above the melting transition of the vortex lattice, where the vortex liquid shows strong short-range order and a large viscosity. Finally, at higher temperatures in the disordered phase, the conductivity is finite, smooth and again nonmonotonically with $k$. This last, monotonic behavior applies in zero magnetic field for the entire disordered phase, i.e., at all temperatures above $T_c$, while in a field the nonmonotonic behavior may occur in a low-temperature portion of the disordered phase.

I. INTRODUCTION

In this paper we explore the nonlocal dc electrical transport properties of type-II superconductors. What is meant by nonlocal here? The general expression connecting the local current density in a material, $J$, to the local electric field $E$ in the linear (Ohmic) regime is

$$J_\mu(r) = \int \sigma_{\mu\nu}(r, r') E_\nu(r') \, dr'.$$

If the conductivity $\sigma(r, r')$ is nonvanishing for $r \neq r'$, then this is nonlocal; the current at $r$ is determined by the field not only at $r$, but at all points $r'$ where $\sigma(r, r') \neq 0$. In a translationally invariant system, the nonlocal conductivity can only be a function of the difference $(r - r')$. Taking the Fourier transform one then obtains $J_\mu(k) = \sigma_{\mu\nu}(k) E_\nu(k)$. All materials exhibit nonlocal transport properties on some length scale. In normal metals, the nonlocal terms are significant only at length scales less than or of the order of the inelastic mean free path. In superconducting materials, however, as the transition is approached from above, the associated correlations can cause nonlocal effects to become important over much longer length scales. It is this phenomenon that we address in this paper.

Vortices are important actors in the nonlocal electrical transport properties in type-II superconductors. This is sometimes more easily described in terms of the nonlocal dc resistivity

$$E_\mu(r) = \int \rho_{\mu\nu}(r, r') J_\nu(r') \, dr'.$$

[Note that the nonlocal resistivity is a linear operator that acts on the full current pattern, $J(r)$, and is, as usual, the inverse of the nonlocal conductivity operator.] A current $J(r')$ pushes on the vortex segments at $r'$ due to the Lorentz and Magnus forces. These vortex segments move, and, due to the continuity and entanglement of vortices as well as the repulsive and attractive forces between parallel and antiparallel (respectively) vortices, they cause vortex segments at $r$ to move. This produces phase slip and electric fields at $r$. Thus vortex dynamics contribute significantly to the nonlocal resistivity, $\rho_{\mu\nu}(r, r')$.

Recent multiterminal transport experiments on YBa$_2$Cu$_3$O$_7$ (YBCO) crystals in the vortex-liquid regime
obtain results that demonstrate nonlocal conductivity on length scales of at least 50 microns. A phenomenological understanding of those experimental results was presented based on a hydrodynamic description of the vortex liquid and its viscosity. In this paper we also examine the vortex-lattice phase and extend the phenomenological study of the vortex-liquid regime to low and zero magnetic field. We then calculate the nonlocal conductivity due to the Gaussian fluctuations above $T_c$ within time-dependent Ginzburg-Landau (TDGL) theory. An outline of the paper is as follows.

Section II considers an ideal, unpinned, defect-free vortex lattice. There the steady-state motion of the vortices is only a rigid-body (flux-flow) motion of the vortex lattice as a whole. The electric field due to this vortex motion is therefore determined only by the total force and torque on the entire vortex lattice. A dc current pattern with wave vector $k \neq 0$ elastically distorts the lattice, but, in linear response, produces no steady motion of the vortices. Thus we find that in this ideal case the Ohmic flux-flow resistivity due to vortex motion is nonzero only at $k = 0$. In both the vortex lattice and Meissner phases there is an additional contribution to the resistivity that is proportional to $k^2$ at small $k$ and does not arise from vortex motion.

In Sec. III we discuss the phenomenology of the vortex-liquid regime. Connectivity, entanglement, and other interactions between nearly parallel vortices then give rise to the vortex-liquid viscosity that reduces their motion in response to nonuniform dc currents, producing a resistivity that has a local maximum for uniform ($k = 0$) currents and is smaller for long-wavelength nonuniform ($k \neq 0$) currents, as in the case of the vortex lattice. However, at high enough temperature and low or zero magnetic field there will be thermally excited vortex lines and loops present with all orientations. Then the connectivity and attraction of nearly antiparallel vortices produce a nonlocal effect of the opposite sign, with the resistivity smallest at $k = 0$. In the disordered phase, the conductivity and the resistivity may be expanded in powers of $k$; these arguments then indicate that the signs of the order $k^2$ terms will depend on temperature and magnetic field.

Section IV sets up the TDGL calculation of the lowest-order (Gaussian approximation) fluctuation contribution to the nonlocal dc conductivity in the disordered phase. For a uniform ($k = 0$) current this Gaussian approximation gives the Aslamazov-Larkin fluctuation conductivity. The calculation for zero magnetic field is carried out in Sec. V. Here the full $k$ dependence can be obtained and the conductivity is found to be a monotonically decreasing function of $k$. Section VI obtains the fluctuation conductivity to order $k^2$ in a magnetic field. At this order, the sign of the $k$ dependence remains the same as for zero magnetic field, at least for longitudinal electric fields. We conclude with a brief summary and discussion in Sec. VII. Appendices A and B are devoted to some technical details of the TDGL calculation.

II. VORTEX LATTICE

Let us first consider the vortex-lattice phase. In dc steady state the total time-averaged force on each portion of the vortex lattice, including drag forces, must vanish, since the lattice is either stationary or moving with a finite steady-state velocity. In linear response to a dc current, the resulting steady-state local force balance equation for the vortex lattice contains four terms: one proportional to and perpendicular to the local current density $J$ (arising from Lorentz and/or Magnus forces), one proportional to and perpendicular to the local vortex velocity $v$ (arising from the Magnus force and/or a perpendicular drag force), one proportional to and parallel to the local vortex velocity (a standard drag force), and one arising from the local elastic distortion of the vortex lattice. Assuming the magnetic induction $\mathbf{B} = B\hat{z}$, so that the vortices are parallel to the $z$ axis, and that the material is isotropic in the $xy$ plane, the steady-state force balance equation is

$$\mathbf{J} \times \mathbf{B} + \gamma_2 n \hat{z} \times \mathbf{v} = \gamma_1 n \mathbf{v} - \frac{\delta H_{\text{elastic}}}{\delta \mathbf{u}} = 0,$$  \hspace{1cm} (2.1)

where

$$H_{\text{elastic}} = \frac{1}{2} \sum_k u_i(-k) \left\{ c_{LL}(k)k_i^2k_j + \delta_{ij}c_{66}(k)k^2 + c_{44}(k)k^2 \right\} u_j(k) \hspace{1cm} (2.2)$$

is the elastic energy of the vortex lattice. $u$ is the local displacement of the vortices away from an ideal, undistorted lattice, and $c_{LL}$, $c_{66}$, and $c_{44}$ are the bulk, shear, and tilt elastic moduli, respectively. Here $\mathbf{v}$ refers to the local velocity of the vortices in the steady state, averaged over time and over a length scale longer than the lattice spacing. We assume the lattice is dislocation free so that $\mathbf{u}$ is well defined, and that there are no vacancies or interstitials so that $\mathbf{v}$ is simply $\partial \mathbf{u}/\partial t$. The areal density $n$ is related to the magnetic induction by $n = B/\phi_0$, with $\phi_0 = h/2e$ the flux quantum. The drag coefficients $\gamma_1$ and $\gamma_2$ are phenomenological parameters.

When the current is spatially uniform, the resulting vortex velocities and displacements are also spatially uniform. The elastic term in (2.1) is then zero and we are left with a balance of the other three terms. Using the Josephson relation for the electric field produced by moving vortices,

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}, \hspace{1cm} (2.3)$$

one finds for the components of the conductivity tensor

$$\sigma_{xx} = \sigma_{yy} = \frac{\gamma_1}{\phi_0 B}; \hspace{1cm} \sigma_{xy} = -\sigma_{yx} = \frac{\gamma_2}{\phi_0 B}; \hspace{1cm} (2.4)$$

$\sigma_{xx}$ is infinite and the other components are zero. The effect is that when a spatially uniform current is applied, the vortex lattice as a whole is pushed across the sample, causing phase slip and therefore dissipation. This is the standard flux-flow resistivity.

Now suppose the current is nonuniform. The elastic term is then nonzero because different parts of the vortex lattice are subject to different forces and this results in elastic strains. If the current produces a nonzero to-
tal force or torque on the vortex lattice, it will move as a (overdamped) rigid body in response, again exhibiting flux-flow resistivity. However, if the spatially averaged current (as well as the total torque applied to the vortex lattice by the current) vanishes, there is no steady-state motion of the vortex lattice. Instead the lattice is statically strained so that everywhere the elastic restoring force locally balances the force due to the nonuniform current. The vortices are not moving, and so no electric field is being produced, although a current is flowing. Therefore, the resistivity due to vortex motion vanishes in linear response to a long-wavelength, static, nonuniform current pattern with zero spatial average. For example, if a dc current is applied in the z direction with variation in the x direction, then the vortices will be pushed along the y direction, producing static shear elastic distortions. Such an example is illustrated in Fig. 1.

Here the dc resistivity due to vortex motion is completely nonlocal, since there is only rigid-body motion of the vortex lattice as a whole, which is determined by the total force (and torque) applied to the entire vortex lattice by the current. Of course, this is a very idealized discussion, since we are neglecting pinning, defects in the vortex lattice (and the resulting plastic motion), and the boundaries of the sample. However, this shows that in this ideal case, the freezing of the vortices into a lattice is indeed a superconducting transition. Although the flux-flow resistivity to a uniform ($k = 0$) current does not vanish when the vortices freeze, the dc flux-flow resistivity to a long-wavelength nonuniform ($k > 0$) current does vanish.

![Diagram of a two-dimensional vortex lattice in a nonuniform current. The magnetic field is parallel to the z axis, normal to the film. The four bold lines running parallel to the y axis are current contacts. A certain amount of current, I, is injected uniformly along the current contact 1 and is withdrawn by the same amount and uniformly along the current contact 3, while 2I is injected uniformly along the current contact 2 and is withdrawn uniformly along the current contact 2. The resulting current density, J and 2J, is uniform along y, but nonuniform along x, as illustrated. The forces on the vortex lattice due to the nonuniform current elastically strain the lattice, producing the shear displacements, u, shown. These displacements are also uniform along y but vary along x. Outside of the outer contacts, J = 0 and u can be taken to be zero uniformly.]

We have examined the response of the vortex lattice to a nonuniform electric field within the time-dependent Ginzburg-Landau (TDGL) equations in the absence of thermal noise, following, e.g., Troy and Dorsey. The above phenomenology for the resistivity due to vortex motion is confirmed, but for nonuniform currents there is an additional contribution to the resistivity which does not arise from vortex motion. This dissipation in a static, but a strained configuration arises directly from the gradient-squared term in the TDGL equations and gives a resistivity proportional to $k^2$ for small wave vector $k$ in both the vortex-lattice and Meissner phases.

To summarize, the resistivity in the ideal vortex-lattice phase is nonzero at $k = 0$ and proportional to $k^2$ for small nonzero $k$. It is therefore discontinuous and nonmonotonic in $k = 0$. We expect that this discontinuity and nonmonotonicity is still present if the lattice contains a nonzero density of vacancies and interstitials: Then the resistivity for $k = 0$ is due to motion of the entire lattice, while for small nonzero $k$ only the defects move, resulting in a lower resistivity.

### III. VORTEX-LIQUID PHENOMENOLOGY

Let us now consider the vortex-liquid regime. This is the regime in which Huse and Majumdar\(^1\) constructed their phenomenological theory for the conductivity measurements of Safar et al.\(^2\) They begin with a force balance equation including three terms: one proportional to the current density $\mathbf{J}$ (arising from Lorentz and/or Magnus forces), one proportional to the average vortex velocity and hence the electric field (arising from drag and Magnus forces), and one proportional to the second spatial derivative of the vortex velocity and therefore the second spatial derivative of the electric field (arising from viscous forces):

$$J_{\mu}(r) = \sigma_{\mu\nu}(0)E_{\nu}(r) - S_{\mu\alpha\beta\nu}\partial_{\alpha}\partial_{\beta}E_{\nu}(r). \tag{3.1}$$

Although this equation was initially motivated by considering only the electric field due to vortex motion, it is more generally just the long-wavelength expansion of the nonlocal Ohm's law (1.1). The nonlocal conductivity to order $k^2$ is then

$$\sigma_{\mu\nu}(k) = \sigma_{\mu\nu}(0) + S_{\mu\alpha\beta\nu}k_{\alpha}k_{\beta}. \tag{3.2}$$

Note that in this paper we call the coefficient of the last term $S$, not $\eta$ as in Huse and Majumdar,\(^1\) in order to avoid confusion with either the Bardeen-Stephen drag coefficient or the vortex-liquid viscosity tensor.

How is $S$ related to the hydrodynamic viscosity tensor of the vortex liquid? This can be answered for the components of $S$ that couple to electric fields in the $xy$ plane using the more detailed force balance equation, similar to that discussed above for the lattice, given in Marchetti and Nelson.\(^2\) Again, $B = B_{\|}^\perp$. Neglecting compressibility as well as vortex segments which are not parallel to the $z$ axis and any hexatic bond-orientational order, the only difference between the dc force balance equation for the liquid and the lattice is that the elastic force is replaced.
by a viscous force:

\[
B(\mathbf{J} \times \mathbf{v})_i + \gamma_2 n(\mathbf{z} \times \mathbf{v})_i - \gamma_1 n v_i + \frac{1}{2} \eta_{ijk}(\partial_j \partial_k v_i + \partial_j \partial_i v_k) = 0. \tag{3.3}
\]

A note on notation: Bardeen and Stephen refer to the coefficient of the drag term as \( \eta \). Here and in Marchetti and Nelson,\(^2\) the drag coefficients are \( \gamma \)'s, and \( \eta \) is the viscosity, which enters as the coefficient of the \( \nabla^2 \) term. Note also that this \( \eta \) involves interactions between vortices, as well as the connectivity and entanglement of vortex lines. If we neglect any dissipation that is not associated with vortex motion, following the same steps as for the lattice, we obtain

\[
\sigma_{xx}(k) = \frac{1}{\phi_0 k_B} \begin{bmatrix} \gamma_1 + \phi_0 \eta_{\alpha\beta} k_\alpha k_\beta \\
\gamma_2 - \phi_0 \eta_{\alpha\beta} k_\alpha k_\beta \end{bmatrix}, \quad \sigma_{xy}(k) = \frac{1}{\phi_0 k_B} \begin{bmatrix} \gamma_1 + \phi_0 \eta_{\alpha\beta} k_\alpha k_\beta \\
\gamma_2 - \phi_0 \eta_{\alpha\beta} k_\alpha k_\beta \end{bmatrix}, \quad \text{etc.} \tag{3.4}
\]

Expressions for the resulting hydrodynamic contributions to \( S \) can be read off from these equations.

However, the approximation of neglecting vortex segments not running parallel to the \( z \) axis is inappropriate in a vortex liquid whose uniform \((k = 0)\) resistivity parallel to the \( z \) axis is nonzero. This resistivity is due to the motion of precisely those vortex segments that do not run parallel to the \( z \) axis. Thus more generally we should consider a vortex liquid containing vortices running in any direction. In a large enough magnetic field and at low enough temperatures, only the field-induced vortices are present, and they are all nearly parallel to each other and to the \( z \) axis. When a uniform current is applied perpendicular to the vortices they all move in the same direction, as in the case of the vortex lattice. However, in a nonuniform \((k = 0)\) current vortices in neighboring regions experience different forces. Unlike in the vortex lattice the resulting shears are not simply balanced by elastic forces, instead the vortices do continue to move past each other. However, their motion is slowed (relative to the case of a uniform current) by the vortex-liquid viscosity arising from connectivity, entanglement, and other vortex-vortex interactions. Slower vortex motion means reduced flow-flow resistivity. Thus the flux-flow resistivity decreases with increasing \( k \) and the conductivity increases. This corresponds to positive viscosities in the above hydrodynamic model and gives, for example, \( S_{\alpha\beta\gamma\delta} > 0 \).

At higher temperatures and in low or zero magnetic field, there will be thermally excited vortex line segments present with all orientations that outnumber the field-induced vortices. Parallel vortices interact repulsively, while antiparallel vortices attract each other. Thus the near neighbors to a given vortex segment are more likely to be antiparallel. Such nearby antiparallel segments can be joined into a vortex loop, or, in a film, can form a bound vortex-antivortex pair. When a current is applied, those segments perpendicular to the current will feel the resulting Magnus/Lorentz force. If the current is in the \( x \) direction, a vortex segment parallel to positive \( \hat{z} \) will be pushed in the negative \( y \) direction while a segment parallel to negative \( \hat{z} \) will be pushed in the positive \( y \) direction. Both motions induce electric fields of the same sign along the \( x \) direction and thus contribute to the flux-flow resistivity. However, the relative motion of such antiparallel vortices is impeded by their being connected or entangled, as well as by the attractive force between antiparallel vortices. In a uniform \((k = 0)\) current the antiparallel segments feel equal and opposite forces from the current, and so this is when their motion is most impeded by connectivity, entanglement, or other interactions. For a nonuniform \((k > 0)\) current, the forces do not cancel and part of the motion generated is "center of mass" instead of relative, and so is not impeded by the interactions. Thus we expect more vortex motion for \( k > 0 \), in this case where it is the relative motion of antiparallel vortices that dominates the resistivity. More vortex motion means a larger electric field, greater resistivity, and lower conductivity. Thus here the conductivity for small \( k \) is maximal for uniform \((k = 0)\) current and decreases with increasing \( k \). This corresponds to negative viscosities in the above hydrodynamic model and gives for example \( S_{\alpha\beta\gamma\delta} < 0 \). Thus we expect that the sign of the nonlocal effect (the \( S_{ij} \)'s) will change as one varies the field and/or temperature in the vortex-liquid regime. This sign change has been confirmed for \( S_{\alpha\beta\gamma\delta} \) in preliminary Monte Carlo simulations of a simple two-dimensional model superconductor.\(^1\)

The above phenomenological arguments deal with the long-wavelength (small \( k \)) behavior. At short wavelengths (large \( k \)) we expect the qualitative behavior is not sensitive to the various phase transitions and distinctions that affect the long-wavelength behavior. In all the regimes where we can obtain the large-\( k \) behavior, namely, the Meissner and vortex lattice phases in TDGL without fluctuations and the treatment below of the fluctuation conductivity in TDGL in the zero-field normal state, we find that the conductivity decreases with increasing \( k \) in the large-\( k \) regime. Thus we expect this remains true throughout the vortex liquid as well. This then suggests that when the \( k \) dependence at small \( k \) changes sign, the conductivity is changing from monotonic in \( k \) (in the higher-temperature, lower-field regime), to nonmonotonic in \( k \) (in the lower-temperature, higher-field regime).

IV. TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS AND LINEAR RESPONSE

In order to calculate the nonlocal conductivity of a superconductor we need to specify the dynamical equations of motion for the superconducting order parameter \( \psi \). We will adopt the simplest such description, the time-dependent Ginzburg-Landau (TDGL) equation:

\[
\Gamma^{-1} \left( \partial_t + i \frac{e^*}{\hbar} \phi \right) \psi = \frac{\hbar^2}{2m} \left( \nabla - i \frac{e^*}{\hbar} A \right)^2 \psi - a \psi - b |\psi|^2 \psi + \zeta, \tag{4.1}
\]

where \( \phi \) is the scalar potential, \( m \) is the effective mass.
of a Cooper pair, $e^* = 2e$ is the charge of a Cooper pair, $a(T) = a_0(T/T_c - 1)$ with $T$ the temperature and $T_c$ the zero-field critical temperature in the absence of the noise term, and $\Gamma$ is the order parameter relaxation rate (taken to be real). Note that we are treating an isotropic superconductor. The stochastic noise term $\zeta(x, t)$ is chosen to have Gaussian white noise correlations, with the two-point correlation function given by

$$\langle \zeta^*(x, t) \zeta(x', t') \rangle = 2\Gamma^{-1} - k_B \theta(x - x') \delta(t - t'), \quad (4.2)$$

with the coefficient being determined by the fluctuation-dissipation theorem.\(^{12}\) We will work in the limit of large Ginzburg-Landau parameter, $\kappa = \lambda / \xi$, where we can neglect the fluctuations in the magnetic field. Thus the vector potential $A$ is static and is simply that due to a uniform magnetic field, and $\mathbf{H} = B$.

As we are interested in the linear response of the system to an applied electric field, we can calculate the conductivity matrix by using the Kubo formula, which expresses the conductivity as the Fourier transform of the current-current correlation function

$$\sigma_{\mu\nu}(k, \omega) = \frac{1}{2k_B T} \int d^d(x - x') \int d(t - t') \psi^*(x - x') - i\omega(t - t')J_\mu(x, t)J_\nu(x', t'). \quad (4.3)$$

Since we are assuming fluctuations in the vector potential, the current which appears in the Kubo formula is the supercurrent,

$$\mathbf{J}_s = \frac{e^*}{2m_i} \left( \mathbf{\psi}^* \nabla \psi - \psi \nabla \psi^* \right) - \frac{(e^*)^2}{m} |\psi|^2 \mathbf{A}, \quad (4.4)$$

and the conductivity is due to superconducting fluctuations; the total conductivity is obtained by adding this contribution to the normal-state conductivity. The validity of the Kubo formula in the context of the TDGL equations (with real $\Gamma$) can be demonstrated.\(^{13}\) However, when $\Gamma$ is complex, the usual form of the Kubo formula may need to be modified.\(^{13}\)

If we make the Gaussian approximation and neglect the cubic term in the TDGL equation, Eq. (4.1), then the current-current correlation function factors into a product of order parameter correlation functions, with the result that

$$\sigma_{\mu\nu}(k) = \frac{1}{2k_B T} \left( \frac{e^*}{2m_i} \right)^2 \int \frac{d\omega}{2\pi} \int d^d(x_1 - x_2) e^{i k \cdot (x_1 - x_2)} \times \left( D_{\mu1} - D_{\mu3} \right) \left( D_{\nu2} - D_{\nu4} \right) C_0(x_2, x_3, \omega) \times C_0(x_1, x_4, \omega))_{\omega = 1, 4 = 2}, \quad (4.5)$$

where $D_\mu = \partial_\mu - ieA_\mu / \hbar$, and

$$C_0(x, x'; \omega) = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \langle \psi(x, t) \psi^*(x', t') \rangle \approx \frac{2k_B T}{\omega} \text{Im} G_0(x, x'; \omega), \quad (4.6)$$

with $G_0(x, x'; \omega)$ the order parameter response function, which is the solution to

$$\left[ -i\Gamma^{-1} \omega - \frac{\hbar^2}{2m} \left( \nabla - \frac{e^*}{\hbar} \right)^2 + a \right] G_0(x, x'; \omega) = \delta^d(x - x'). \quad (4.7)$$

Within the Gaussian approximation we have also obtained the conductivity directly from the equation of motion (4.1), as a check on the Kubo formula calculation. Thus for general $k$ in zero magnetic field and to order $k^2$ in a magnetic field we have confirmed that the fluctuation contribution to the dc conductivity matrix is indeed symmetric and given by (4.3). Note that the Hall conductivity is zero.\(^{14}\)

The Gaussian approximation is valid in a region well above (as defined by the Ginzburg criterion) the superconducting transition, where there are only small-amplitude fluctuations of the order parameter. When fluctuations are large, the nonlinear terms become important and the Gaussian approximation no longer applies. It is precisely in this strong-fluctuation regime where a description in terms of fluctuating vortices becomes appropriate. This is the vortex-liquid regime, for which this calculation is inappropriate. However, the low-temperature end of these Gaussian results may exhibit some of the properties of the high-temperature behavior of the technically more difficult vortex-liquid regime. In principle, we could use the Hartree approximation\(^{14}\) to include a partial contribution from the nonlinear terms. However, because the Hartree approximation only renormalizes the critical temperature and thus will not change the signs or any other qualitative properties of $S_{ijkl}$, in this paper we will only focus on the Gaussian approximation.

V. ZERO MAGNETIC FIELD

Before tackling the technically difficult task of calculating the nonlocal conductivity tensor in an applied magnetic field, we will first calculate the nonlocal conductivity in zero magnetic field in the Gaussian approximation. In this case the system is translationally invariant, and after Fourier transforming the Kubo formula, Eq. (4.5), we obtain

$$\sigma_{\mu\nu}(k) = \frac{2}{k_B T} \left( \frac{e^*}{2m} \right)^2 \int \frac{d\omega}{2\pi} \int \frac{dp}{(2\pi)^d} \int \frac{dP_m}{P_m} \int \frac{dP_{\nu}}{d\nu} \times C_0(p + k/2, \omega) C_0(p - k/2, \omega), \quad (5.1)$$

where the correlation function is

$$C_0(k, \omega) = \frac{2k_B T \Gamma^{-1}}{\omega^2 + (\hbar^2 k^2/2m + \alpha)^2}. \quad (5.2)$$

For $d < 4$ the integral is ultraviolet convergent, and so we do not need a cutoff. After substituting the correlation function into Eq. (5.1), performing the frequency integral, and scaling the momenta by the correlation length $\xi = \hbar / \sqrt{2m|a|}$, we obtain

$$\sigma_{\mu\nu}(k) = \sigma(0) F_{\mu\nu}(k\xi), \quad (5.3)$$
where the scaling function $F_{\mu\nu}(x)$ is normalized so that $F_{\mu\nu}(0) = \delta_{\mu\nu}$, and where the $k = 0$ conductivity is\(h^{18}\)

$$\sigma(0) = k_B T \left( \frac{m (e^*)^2}{\hbar \Gamma} \right) \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} \xi^{4-d}. \quad (5.4)$$

The calculation of the scaling functions $F_{\mu\nu}(k\xi)$ is rather complicated, and the details are relegated to Appendix A. The conductivity can be decomposed into transverse and longitudinal components

$$\sigma_{\mu\nu}(k) = \sigma(0)[F^T(k\xi)P^T_{\mu\nu} + F^L(k\xi)P^L_{\mu\nu}], \quad (5.5)$$

where we have introduced the transverse and longitudinal projection operators

$$P^T = \frac{k_\mu k_\nu}{k^2}, \quad P^L = \frac{k_\mu k_\nu}{k^2}. \quad (5.6)$$

A steady-state electric field is purely longitudinal (it is the gradient of the scalar potential), and so only the longitudinal part of the conductivity enters in determining a dc current pattern. In the Gaussian approximation, the transverse and longitudinal scaling functions can be obtained in closed form (see Appendix A) and are plotted in Figs. 2 and 3. Both functions decrease monotonically with increasing $k$. As shown in Appendix A, for large $x$, $F^T(x) \sim c^{T}_{d}(x)^{(4-d)/2}$, with $c^{T}_{d}$ universal constants (there are logarithmic corrections in two dimensions). Expanding the scaling functions to $O(k^2)$, we obtain

$$\sigma(0)F^T(k\xi) = \sigma(0) + S^T k^2 + \cdots, \quad (5.7)$$

with

$$S^T = -\frac{5}{48} (4-d) \xi^2, \quad S^L = -\frac{1}{16} (4-d) \xi^2. \quad (5.8)$$

The full $S$ tensor can be written compactly as

$$S_{\mu\alpha\beta\nu} = S^T \delta_{\mu\nu} \delta_{\alpha\beta} + \frac{1}{2} (S^L - S^T)(\delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha}). \quad (5.9)$$

Finally, after Fourier transforming back to real space, we find for the current

$$J(x) = \sigma(0)E(x) + S^T \nabla \times \nabla \times E(x) \times + S^L \nabla \cdot E(x) + \cdots. \quad (5.10)$$

Note that $S^L$ (which is equal to, e.g., $S_{zzzz}$) is negative here in the Gaussian approximation, and is argued above (Sec. III) to remain negative in the vortex liquid in the critical regime just above $T_c$ at $H = 0$. The general scaling form (5.5) presumably also remains valid in the critical regime, but with quantitatively different scaling functions. However, the qualitative behavior of the conductivity—that it is maximal for $k = 0$ and falls off as a power of $k$ for large $k\xi$—should be the same in the nontrivial critical regimes for $H = 0$. Thus there is no sign here that anything dramatic occurs in the nonlocal conductivity at the Ginzburg crossover from mean-field to nontrivial critical behavior in zero magnetic field.

**VI. NONZERO MAGNETIC FIELD**

In this section we will calculate the nonlocal conductivity in the Gaussian approximation in a uniform applied magnetic field $H = H^2$. We will work in the Landau gauge $A = (0, Hx, 0)$. The response function is obtained as an eigenfunction expansion in a harmonic oscillator basis set in the $x$ direction—the Landau-level expansion. Using Eq. (4.7), we then obtain for the correlation function in a mixed representation

$$C_0(x, x'; k_y, k_z, \omega) = 2k_B \Gamma^{-1} \sum_{n=0}^{\infty} \frac{u_n(x - x_0)u_n(x' - x_0)}{(\omega/\Gamma)^2 + \epsilon_n^2}, \quad (6.1)$$

where the oscillator functions are

$$u_n(x) = \left( \frac{1}{2^n n! \sqrt{n!}} \right)^{1/2} e^{-x^2/2\lambda^2} H_n(x/L_H), \quad (6.2)$$
with the energies
\[ \epsilon_{n,k_z} = \frac{\hbar^2 k_z^2}{2m} + \hbar \omega_c \left( n + \frac{1}{2} \right) + a. \] (6.3)

In the above we have introduced the magnetic length \( l_H = (\hbar/e^*H)^{1/2} \), the orbit center coordinate \( x_0 = l_H^2 k_y \), and the cyclotron frequency \( \omega_c = e^*H/m \).

**A. \( J, E \) perpendicular to \( H \)**

In this subsection we calculate the nonlocal conductivity when the current and electric field are in the \( x-y \) plane. Given the symmetries of the system, in this geometry at order \( k^2 \) there are only three independent coefficients to be calculated: \( \sigma_{yy}, \sigma_{xx}, \) and \( \sigma_{xy} \). To obtain these it is only necessary to calculate \( \sigma_{yy}(k) \). Rotational symmetry about the \( z \) axis gives \( \sigma_{zz} = \sigma_{yy} \), \( \sigma_{xx} = \sigma_{yy}, \) \( \sigma_{axx} = \sigma_{axy} \), etc., and \( \sigma_{zxy} = \sigma_{yy} - \sigma_{xx} \). Due to the symmetries and the absence of a Hall effect, all \( S_{ijkl} \) with unpaired indices (e.g., \( \sigma_{yy} \)) vanish in this calculation.

To calculate \( \sigma_{yy}(k) \), we carry out the differentiations indicated in Eq. (4.5) (noting that \( D_y = \partial_y - i eH \bar{z}/\hbar \)) using the correlation function in Eq. (6.1). The frequency integral is then easily performed; after scaling \( x \) and \( x_0 \) by the magnetic length \( l_H \), and rescaling the external momenta in the \( x-y \) plane by \( l_H \) so that \( (\vec{k}_x, \vec{k}_y) = c \frac{1}{l_H} H, k_y x_H \) so that \( (k_x, k_y) = (k_x l_H, k_y l_H) \), we obtain

\[
\sigma_{yy}(k) = k_B T \Gamma^{-1} (e^*)^2 \left( \frac{e^*H}{m} \right)^2 \frac{1}{2\pi} \sum_{m,n=0}^{\infty} I_{mn}(\vec{k}_x, \vec{k}_y) \\
\times \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \epsilon_m(p_z + k_z/2) \epsilon_n(p_z - k_z/2) \left[ \epsilon_m(p_z + k_z/2) + \epsilon_n(p_z - k_z/2) \right],
\]

(6.4)

where

\[
I_{mn}(\vec{k}_x, \vec{k}_y) = \int_{-\infty}^{\infty} \frac{dx_1}{2\pi} \frac{dx_2}{2\pi} e^{ik_z(x_1 - x_2)} \int_{-\infty}^{\infty} dx_0 (x_0 - x_1)(x_0 - x_2) u_n(x_2 - x_0 - \vec{k}_y/2) \\
\times u_m(x_1 - x_0 - \vec{k}_y/2) u_m(x_1 - x_0 + \vec{k}_y/2) u_m(x_2 - x_0 + \vec{k}_y/2).
\]

(6.5)

After shifting the integration variables, this integral can be transformed into a single integral:

\[
I_{mn}(\vec{k}_x, \vec{k}_y) = |A_{mn}^{(1)}(\vec{k}_x, \vec{k}_y)|^2,
\]

(6.6)

where

\[
A_{mn}^{(1)}(\vec{k}_x, \vec{k}_y) = \int_{-\infty}^{\infty} dy e^{-ik_y y} u_n(y + \vec{k}_y/2) u_m(y - \vec{k}_y/2).
\]

(6.7)

Some simplification also occurs in the \( p_z \) integral, if we first rescale \( p_z \) by the zero-temperature correlation length \( \xi(0) = (\hbar^2/ma_0)^{1/2} \), and introduce \( \bar{k}_z = k_z \xi(0) \). Then we find

\[
\sigma_{yy}(k) = \frac{m \xi(0)}{8\pi \hbar^2 \Gamma T} \sum_{m,n=0}^{\infty} |A_{mn}^{(1)}(\vec{k}_x, \vec{k}_y)|^2 B_{mn}^{(1)}(\bar{k}_z),
\]

(6.8)

where we have introduced the thermal length \( \Lambda_T = \phi_0^2/16\pi^2k_B T \), the flux quantum \( \phi_0 = 2\pi\hbar/e^* \), and the scaled magnetic field \( h = H/H_{c2}(0) \), with \( H_{c2}(0) = \phi_0/2\pi \xi^2(0) \) the zero-temperature critical field. The integral \( B_{mn}^{(1)} \) is given by

\[
B_{mn}^{(1)}(\bar{k}_z) = 4\hbar^2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{[(p + \bar{k}_z/2)^2 + \mu_m][(p - \bar{k}_z/2)^2 + \mu_n]} [p^2 + (\bar{k}_z/2)^2 + \mu_{m+n}/2],
\]

(6.9)

where

\[
\mu_m = \epsilon_H + 2hm, \quad \epsilon_H = (T/T_c) - 1 + h.
\]

(6.10)

Within mean-field theory the transition to the flux lattice state occurs at \( \epsilon_H = 0 \).

We now perform a long-wavelength expansion of the conductivity. The expansion of \( A_{mn}^{(1)} \) is most easily carried out by first noting that it can be written in a compact operator form (with \( \hat{p} = \hat{\frac{1}{i}} \hat{\phi} \) the momentum operator)

\[
A_{mn}^{(1)}(\vec{k}_x, \vec{k}_y) = \langle m | e^{-\frac{1}{2} \hat{k}_x \hat{p} - \frac{1}{2} \hat{k}_y \hat{y} - \frac{1}{2} \hat{k}_z \hat{y} e^{-\frac{1}{2} \hat{k}_z \hat{p}} | n \rangle \\
= e^{\frac{1}{2} \hat{k}_x \hat{p}} \langle m | e^{-\frac{1}{2} (\hat{k}_x \hat{y} + \hat{k}_y \hat{p}) - \frac{1}{2} (\hat{k}_z \hat{y} + \hat{k}_z \hat{p})} | n \rangle,
\]

(6.11)
where we have used \( \hat{y}, \hat{p} = i \) to obtain the last line. Expanding for small \((\hat{k}_x, \hat{k}_y)\), we find (neglecting the multiplicative phase factor)

\[
A^{(1)}_{mn}(\hat{k}_x, \hat{k}_y) = \langle m | \hat{y} | n \rangle - i \langle m | \hat{y}^2 | n \rangle \hat{k}_x - \frac{i}{2} \langle m | \hat{y} \hat{p} + \hat{p} \hat{y} | n \rangle \hat{k}_y - \frac{i}{2} \langle m | \hat{y}^3 | n \rangle \hat{k}_x^2 - \frac{1}{8} \langle m | \hat{y}^2 \hat{p} + 2 \hat{p} \hat{y} \hat{p} + \hat{p}^2 \hat{y} | n \rangle \hat{k}_y^2 - \frac{1}{8} \langle m | 3 \hat{y}^2 \hat{p} + 2 \hat{p} \hat{y} \hat{p} + 3 \hat{p}^2 \hat{y} | n \rangle \hat{k}_x \hat{k}_y + O(k^3).
\] (6.12)

The oscillator matrix elements are tabulated in Appendix B. After squaring, we find

\[
|A^{(1)}_{mn}(\hat{k}_x, \hat{k}_y)|^2 = \frac{1}{2} [n \delta_{m,n-1} + (n + 1) \delta_{m,n+1}] + [(n - 1) n \delta_{m,n-2} - 3 n^2 \delta_{m,n-1} \\
+ (2n + 1) \delta_{m,n-3} + (n + 1)^2 \delta_{m,n+1} + (n + 1)(n + 2) \delta_{m,n+2}] (\hat{k}_x/2)^2 \\
+ [(n - 1)n \delta_{m,n-2} - n^2 \delta_{m,n-1} - (n + 1)^2 \delta_{m,n+1} + (n + 1)(n + 2) \delta_{m,n+2}] (\hat{k}_y/2)^2 + O(\hat{k}_x, \hat{k}_y).
\] (6.13)

The long-wavelength expansion of \( B^{(1)}_{mn}(\hat{k}_z) \) is discussed in Appendix B. After substituting these expansions, Eqs. (6.13) and (B10), into the expression for the conductivity, Eq. (6.8), and returning to conventional units, we obtain the \( k = 0 \) conductivity

\[
\sigma_{yy}(0) = \frac{m \xi(0)}{8 \pi \hbar^2 \Gamma \Lambda T \hbar^{1/2}} \sum_{n=0}^{\infty} \frac{(n + 1)}{(\alpha + 2n)^{1/2}} - \frac{2}{(\alpha + 2n + 1)^{1/2}} + \frac{1}{(\alpha + 2n + 2)^{1/2}},
\] (6.14)

which agrees with previous results.\(^{17-19,14}\) We have defined the scaling variable \( \alpha = e H / h \), which measures the temperature distance from the mean-field \( H_0 \) line in units proportional to the magnetic field. Thus \( \alpha \) runs from zero on the \( H_0 \) line at nonzero field to infinity at \( H = 0 \) in the normal state above \( T_c \). Note that the sum here (and in many of the results below) is a scaling function that depends only on this one parameter, \( \alpha \). This type of sum may be written in a more compact form by using the integral representation for the function \( \Phi(z, s, v) \).\(^{20}\) After resummation, they are the Laplace transformations of certain functions, and so we define

\[
\sigma_{ij}(0) = \sigma(0) (\alpha - 1)^{-1/2} \int_0^\infty \frac{dt}{\sqrt{\pi}} \exp(-\alpha t) \tilde{\sigma}_{ij}(0),
\] (6.15)

\[
S_{ijkl} = \sigma(0) \xi^2(T) (\alpha - 1)^{-3/2} \int_0^\infty \frac{dt}{\sqrt{\pi}} \exp(-\alpha t) \tilde{S}_{ijkl},
\] (6.16)

where we have expressed the prefactor \( m \xi(0)/8 \pi \hbar^2 \Gamma \Lambda T \) in terms of \( \alpha \), the zero-field coherence length \( \xi(T) \), and \( \sigma(0) \), the \( k = 0, H = 0 \) conductivity at temperature \( T \). We find

\[
\tilde{\sigma}_{yy}(0) = \frac{4t^{-1/2}}{[1 + \exp(-t)]^2}.
\] (6.17)

We also obtain

\[
S_{yyyy} = \frac{m \xi(0)}{8 \pi \hbar^2 \Gamma \Lambda T \hbar^{1/2}} \left( \frac{1}{2} \right)^2 \sum_{n=0}^{\infty} \left[ \frac{(n + 1)(3n + 2)}{2(\alpha + 2n)^{1/2}} + \frac{4(n + 1)^2}{(\alpha + 2n + 1)^{1/2}} - \frac{(n + 1)(3n + 4)}{(\alpha + 2n + 2)^{1/2}} + \frac{(n + 1)(n + 2)}{2(\alpha + 2n + 4)^{1/2}} \right],
\] (6.18)

\[
S_{yyyy} = \frac{-t^{-1/2}[1 - \exp(-t)]}{[1 + \exp(-t)]^3},
\]

\[
S_{zyzy} = \frac{m \xi(0)}{8 \pi \hbar^2 \Gamma \Lambda T \hbar^{1/2}} \left( \frac{1}{2} \right)^2 \sum_{n=0}^{\infty} \left[ \frac{3}{4} \frac{(2n + 1)^2}{(\alpha + 2n)^{1/2}} - \frac{(n + 1)(11n + 10)}{2(\alpha + 2n)^{1/2}} + \frac{12(n + 1)^2}{(\alpha + 2n + 1)^{1/2}} - \frac{(n + 1)(7n + 8)}{(\alpha + 2n + 2)^{1/2}} + \frac{(n + 1)(n + 2)}{2(\alpha + 2n + 4)^{1/2}} \right],
\]

\[
S_{zyzy} = t^{-1/2} \left[ \frac{(t^2 - 5) \exp(-4t) + 12 \exp(-3t)}{[1 - \exp(-2t)]^3} + \frac{(6t^2 - 14) \exp(-2t) + 12 \exp(-t) + t^2 - 5}{[1 - \exp(-2t)]^3} \right],
\] (6.19)
\[ S_{yzz} = \frac{m\xi(0)}{8\pi\hbar^2\Gamma_\Lambda h^{3/2}} \left( \frac{\xi(0)}{2} \right)^2 \sum_{n=0}^{\infty} \left\{ \frac{(n + 1)}{(\alpha + 2n + 1)^{3/2}} + 4(n + 1) \left[ \frac{1}{(\alpha + 2n + 2)^{1/2}} - \frac{1}{(\alpha + 2n)^{1/2}} \right] \right\} \]

\[ -12(n + 1) \left[ (\alpha + 2n)^{1/2} - 2(\alpha + 2n + 1)^{1/2} + (\alpha + 2n + 2)^{1/2} \right] \]

\[ \tilde{S}_{yzz} = \frac{2t^{-3/2} \left[ (2t + 3) \exp(-2t) + (t^2 - 6) \exp(-t) + 3 - 2t \right]}{[1 - \exp(-2t)]^2} \]  

(6.20)

An advantage of the integral representations is that one can determine the signs much more easily. We find that for \( t > 0 \), \( \tilde{S}_{ijkl} \) and \( \tilde{\sigma}_{ij} \) are either always positive \( \{\tilde{S}_{yzz}, \tilde{\sigma}_{yy}(0), \text{and } \tilde{\sigma}_{zz}(0)\} \) or always negative \( \{\tilde{S}_{yppp}, \tilde{S}_{zzzz}, \tilde{S}_{zyp}, \tilde{S}_{zyz} + 2\tilde{S}_{zyy}, \tilde{S}_{yzz} \} \). The only exception is \( \tilde{S}_{yzz} \) which changes sign at about \( t = 1.45 \). As a result, \( \tilde{S}_{yzz} \) changes sign at about \( \alpha = 1.2 \) \( \{\tilde{S}_{yzz} \) becomes positive for \( \alpha \approx 1.2 \).

The zero-field limit \( (\alpha \to \infty) \) can also be easily obtained in the integral representations. Simply by redefining \( \alpha \) as \( x \) in (6.15) and (6.16) and then keeping only the leading-order terms of the power series in \( x \) of the integrands, we can recover the zero-field results of the previous section.

**B. J, E parallel to H**

In this geometry we calculate \( \sigma_{zz}(k) \). The calculation of \( \sigma_{zz} \) closely parallels the calculation of \( \sigma_{yy} \), and so we will only outline the results. First, we can express the conductivity as

\[ \sigma_{zz}(k) = \frac{m\xi(0)}{32\pi\hbar^2\Gamma_\Lambda} \sum_{m,n=0}^{\infty} |A_{mn}^{(2)}(\kappa_x, \kappa_y)|^2 B_{mn}^{(2)}(\kappa_z), \]  

(6.21)

where

\[ A_{mn}^{(2)}(\kappa_x, \kappa_y) = \int_{-\infty}^{\infty} dy e^{-i\kappa_y y} u_m(y + \kappa_y/2) u_n(y - \kappa_y/2) \]

\[ = e^{\frac{1}{2} \kappa_y \kappa_y} \langle m|e^{i(k_x y + k_y y)}|n \rangle \]  

(6.22)

and

\[ B_{mn}^{(2)}(\kappa_z) = 16 \int_{-\infty}^{\infty} dp \frac{p^2}{2\pi} \left[ (p + \kappa_z/2)^2 + \mu_m \right] \left[ (p - \kappa_z/2)^2 + \mu_n \right] [p^2 + (\kappa_z/2)^2 + \mu_{m+n}]. \]  

(6.23)

Performing the long-wavelength expansion with the help of the results in Appendix B, we have for \( A_{mn}^{(2)}(k_x, k_y) \)

\[ |A_{mn}^{(2)}(\kappa_x, \kappa_y)|^2 = \delta_{m,n} + \frac{1}{2} |n| \delta_{m,n-1} - (2n + 1) \delta_{m,n} \]

\[ + (n + 1) \delta_{m,n+1} (k_x^2 + k_y^2) \]

\[ + O(k_x^4, k_y^4, k_x^2 k_y^2). \]  

(6.24)

Combining this with the long-wavelength expansion of \( B_{mn}^{(2)}(k_z) \), we find for the conductivity

\[ \sigma_{zz}(0) = \frac{m\xi(0)}{32\pi\hbar^2\Gamma_\Lambda h^{1/2}} \sum_{n=0}^{\infty} \frac{1}{(\alpha + 2n)^{3/2}}, \]

(6.25)

which again agrees with previous results,\(^{17-19,14}\) and

\[ S_{yzz} = -\frac{3m\xi^3(0)}{512\pi^2\hbar^3\Gamma_\Lambda h^{3/2}} \sum_{n=0}^{\infty} \frac{1}{(\alpha + 2n)^{5/2}}, \]

(6.27)
C. J perpendicular to H, E parallel to H

We now need to calculate \( \sigma_{yz}(k) \). Following the same steps as in the previous two sections, we find for the conductivity

\[
\sigma_{yz}(k) = -\frac{m\xi(0)}{8\pi^2 h^2 \Gamma T} \frac{1}{h^{1/2}} \sum_{m,n=0}^{\infty} \frac{p}{(p+\tilde{k}_z/2)^2 + \mu_m} \left[ \frac{1}{(p-\tilde{k}_z/2)^2 + \mu_n} \right] \times A^{(1)}_{mn}(\tilde{k}_x, \tilde{k}_y) A^{(2)}_{mn}(\tilde{k}_x, \tilde{k}_y) B^{(3)}_{mn}(\tilde{k}_z),
\]

where

\[
B^{(3)}_{mn}(\tilde{k}_z) = 4\hbar^2 \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{p}{[(p+\tilde{k}_z/2)^2 + \mu_m][(p-\tilde{k}_z/2)^2 + \mu_n][p^2 + (\tilde{k}_z/2)^2 + \mu_{m+n}/2]}.
\]

Expanding the \( A_{mn} \)'s using the results of the previous sections, we find

\[
A^{(1)}_{mn}(\tilde{k}_x, \tilde{k}_y) A^{(2)}_{mn}(\tilde{k}_x, \tilde{k}_y) = i[n\delta_{m,n-1} + (2n+1)\delta_{m,n} + (n+1)\delta_{m,n+1}] (\tilde{k}_z/2)
- [n\delta_{m,n-1} + (n+1)\delta_{m,n+1}] (\tilde{k}_y/2) + O(\tilde{k}_x^2, \tilde{k}_y^2, \tilde{k}_z^2).
\]

Combining this result with the long-wavelength expansion of \( B^{(3)}_{mn} \) in Appendix B, we obtain

\[
S_{yyzz} = \frac{m\xi(0)}{32\pi^2 h^2 \Gamma T} \frac{1}{\hbar} \sum_{n=0}^{\infty} \left[ (n+1) \left( \frac{1}{\mu_n^{1/2}} - \frac{1}{\mu_{n+1}^{1/2}} \right) + \frac{4(n+1)}{\hbar} \left( \mu_n^{1/2} + \mu_{n+1}^{1/2} - 2\mu_{n+1/2}^{1/2} \right) \right],
\]

\[
\tilde{S}_{yyzz} = \frac{t^{-3/2} [(t+2) \exp(-t) + t - 2]}{[1 - \exp(-2t)][1 + \exp(-t)]}.
\]

In zero field, \( S_{yyzz} \) approaches \((S^L - S^T)/2\) which is positive.

Finally, we find that \( \sigma_{xy}(k_x, k_y, k_z) = \sigma_{yz}(-k_x, k_y, k_z) \), so that \( S_{zzzz} = S_{yyzz} \).

D. General geometry with longitudinal electric field

In a truly dc steady state the electric field must be purely longitudinal. Thus let us consider general wave vector \( k \) and ask what the current is in linear response to a longitudinal electric field. Without loss of generality, we can take \( k = k_y \hat{y} + k_z \hat{z} \), \( E_y = E k_y/k \), and \( E_z = E k_z/k \). To order \( k^2 \), the resulting current is

\[
J_y = E k_y [\sigma_{yy}(0) + S_{yyyy} k^2_y + (S_{yyzz} + 2S_{yyzz}) k_z^2]/k,
\]

\[
J_z = E k_z [\sigma_{zz}(0) + S_{zzzz} k^2_z + (S_{zyzz} + 2S_{zyzz}) k_y^2]/k,
\]

and, in the absence of a Hall effect, \( J_x = 0 \). Thus we see that in this geometry, the nonlocal effect here in the Gaussian approximation is always of the sign such that the conductivity for a longitudinal electric field is reduced as \( k \) moves away from 0. This is of the opposite sign from what we argue above occurs in the vortex-liquid regime at lower temperatures.

VII. DISCUSSION AND CONCLUSIONS

In this paper we have examined the wave-vector-dependent dc conductivity \( \sigma(k) \) of a type-II superconductor in various regimes, using phenomenological arguments and the TDGL equation. There appear to be at least four qualitatively different regimes of behavior for \( \sigma(k) \): First, in the Meissner phase the conductivity is finite at \( k = 0 \) and monotonically decreasing with increasing \( k \). This behavior should also apply in the pinned vortex-lattice and vortex-glass phases, where there is no vortex motion in linear response to a uniform applied current. Second, in an ideal, unpinned vortex-lattice phase, the conductivity is discontinuous at \( k = 0 \), taking on the finite, flux-flow value at \( k = 0 \) due to vortex motion, but varying as \( k^{-2} \) for small, positive \( k \), where there is no vortex motion. Here the conductivity is still a monotonically decreasing function of \( k \) for \( k > 0 \), but is now nonmonotonic when \( k = 0 \) is included. Third, for all temperatures above \( T_c \) in zero magnetic field and for sufficiently high temperatures in small nonzero magnetic fields, the qualitative behavior seen in the above Gaussian-order TDGL calculation applies. There the conductivity is finite and maximal for \( k = 0 \) and is smooth and monotonically decreasing with increasing \( k \). As argued on a phenomenological level in Sec. III above, this behavior should apply in the vortex liquid in a low-field regime where the dissipation is dominated by the spontaneous, thermally excited vortices rather than the field-induced vortices. Last, phenomenological arguments suggest that in the vortex-liquid regime at sufficiently low temperatures and high fields the conductivity is instead a nonmonotonic function of \( k \): At small \( k \) the conductivity increases with \( k \) due to the large vortex-liquid viscosity that impedes nonuniform motion of the vortex liquid. However, at larger \( k \), where vortex motion is not the dominant effect in determining the conductivity, the more microscopic behavior of a conductivity that decreases with increasing \( k \) prevails, as it does at large \( k \) in all regimes. It is this last vortex-liquid regime that is the least accessible theoretically, because it is a strong thermal fluctuation regime that does not exist in a mean-field or weak-fluctuation treatment of Ginzburg-Landau theory. Thus the qualitative behavior described for the first three regimes can
be obtained directly from the TDGL equations, while the theoretical support for the description of the last, vortex-liquid regime is, at this time, purely phenomenological.

What other experiments might be done to probe the $k$ dependence of the transport properties? In a transport experiment one has access only to the surface of the sample. The voltage contacts can measure the electric field parallel to the surface and the current contacts can set $\mathbf{V} \cdot \mathbf{J}$ at the surface. For a bulk sample, this means one must rely on modeling to deduce what is going on inside the sample. However, in a film geometry, the entire sample is surface, and so one could, in principle, have much more complete measurements and control. With modern microfabrication techniques, it seems a study that probes down to micron or shorter length scales should be feasible. We pose this as a future experimental challenge.

\[
F_{\mu\nu}(\mathbf{k}) = \frac{4(4\pi)^{d/2}}{\Gamma(2 - d/2)} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2 + (k/2)^2 + \mathbf{p} \cdot \mathbf{k} + 1} [p^2 + (k/2)^2 - \mathbf{p} \cdot \mathbf{k} + 1][p^2 + (k/2)^2 + 1]. \tag{A1}
\]

In order to simplify the integrals we use the Feynman parametrization to first combine the first two terms in the denominator, and then once again to fold in the third term, with the result that

\[
\int \frac{1}{[p^2 + (k/2)^2 + \mathbf{p} \cdot \mathbf{k} + 1][p^2 + (k/2)^2 - \mathbf{p} \cdot \mathbf{k} + 1][p^2 + (k/2)^2 + 1]} = 2 \int_0^1 dx \int_0^1 dy \frac{y}{[p^2 + (k/2)^2 + 1 + (2x - 1)y\mathbf{p} \cdot \mathbf{k}^2]^3}. \tag{A2}
\]

We then substitute this result into Eq. (A1), change variables in the momentum integral to $q = \mathbf{p} + (2x - 1)y(k/2)$ to eliminate the terms linear in $k$, and perform the $d$-dimensional momentum integral. We are then left with

\[
F_{\mu\nu}(\mathbf{k}) = F^T(\mathbf{k}) P_{\mu\nu}^T + F^L(\mathbf{k}) \frac{k_{\mu} k_{\nu}}{k^2}, \tag{A3}
\]

where

\[
F^T(\mathbf{k}) = 2 \int_0^1 dw \int_0^1 dy \frac{y}{[1 + (k/2)^2(1 - w^2 y^2)]^{2-d/2}}, \tag{A4}
\]

\[
F^L(\mathbf{k}) = F^T(\mathbf{k}) + (2 - d/2)k^2 \int_0^1 dw \times \int_0^1 dy \frac{w^2 y^3}{[1 + (k/2)^2(1 - w^2 y^2)]^{3-d/2}}, \tag{A5}
\]

and where we have changed variables to $w = 2x - 1$. The integrals on $y$ can be performed, and the remaining integrals on $w$ can be simplified by integrating by parts. We finally obtain

\[
F^T(\mathbf{k}) = 2 \int_0^1 dw \frac{1}{[1 + (k/2)^2(1 - w^2)]^{d/2-2}} - \frac{2}{(d - 2)} \frac{1 + (k/2)^2 [d/2 - 1]}{(k/2)^2}, \tag{A6}
\]

\[
F^L(\mathbf{k}) = \frac{\ln[1 + (k/2)^2]}{(k/2)^2}. \tag{A11}
\]

Plots of these scaling functions are shown in Figs. 2 and 3. For small $k$ these expressions have the following expansions for general dimension $d$:

\[
F^T(\mathbf{k}) = 1 - \frac{5}{48} (4 - d)k^2 + O(k^4), \tag{A12}
\]

\[
F^L(\mathbf{k}) = 1 - \frac{1}{16} (4 - d)k^2 + O(k^4). \tag{A13}
\]
For large $\tilde{k}$ and $d \neq 2$, we have

$$F^T(\tilde{k}) \sim c_d^T(\tilde{k}/2)^{-d}, \quad F^L(\tilde{k}) \sim c_d^L(\tilde{k}/2)^{-d},$$

(A14)

where the constants are functions of dimension $d$, given by

$$c_d^T = 2 \int_0^1 dw \left( 1 - w^2 \right)^{d/2 - 2} - \frac{2}{d - 2}, \quad c_d^L = \frac{2}{d - 2},$$

(A15)

with $c_d^T = \pi - 2$. For large $\tilde{k}$ and $d = 2$,

$$F^T(\tilde{k}) \sim \frac{2 \sqrt{2} \ln(\tilde{k}/2)}{(\tilde{k}/2)^2}, \quad F^L(\tilde{k}) \sim 2 \frac{\ln(k/2)}{(k/2)^2}.$$  \hspace{1cm} (A16)

The large-$\tilde{k}$ behavior agrees with the result of the scaling theory discussed in Sec. III, up to some logarithmic corrections in two dimensions.

**APPENDIX B: EXPANSIONS FOR INTEGRALS IN A MAGNETIC FIELD**

In this appendix we will include some of the details of the calculation of the viscosities in a magnetic field. First, we simply list some of the harmonic oscillator matrix elements which are used to evaluate the integrals $A^{(1)}_{mn}$ and $A^{(2)}_{mn}$:

$$\langle m|\hat{y}|n\rangle = \frac{1}{\sqrt{2}} \left[ \sqrt{n} \delta_{m,n-1} + \sqrt{n+1} \delta_{m,n+1} \right], \hspace{1cm} (B1)$$

$$\langle m|\hat{y}^2|n\rangle = \frac{1}{2} \left[ \sqrt{(n-1)n} \delta_{m,n-2} + (2n+1) \delta_{m,n} + \sqrt{(n+1)(n+2)} \delta_{m,n+2} \right], \hspace{1cm} (B2)$$

$$\langle m|\hat{y}^3|n\rangle = \frac{1}{2\sqrt{2}} \left[ \sqrt{(n-2)(n-1)n} \delta_{m,n-3} + 3n^{3/2} \delta_{m,n-1} - 3(n+1)^{3/2} \delta_{m,n+1} + \sqrt{(n+1)(n+2)(n+3)} \delta_{m,n+3} \right], \hspace{1cm} (B3)$$

$$\langle m|\hat{y}^4|n\rangle = \frac{1}{\sqrt{2}} \left[ -\sqrt{(n-2)(n-1)n} \delta_{m,n-3} + n^{3/2} \delta_{m,n-1} - (n+1)^{3/2} \delta_{m,n+1} + \sqrt{(n+1)(n+2)(n+3)} \delta_{m,n+3} \right], \hspace{1cm} (B4)$$

$$\langle m|\hat{y}^5|n\rangle = \frac{1}{2\sqrt{2}} \left[ \sqrt{(n-3)(n-2)(n-1)n} \delta_{m,n-4} + 3n^{3/2} \delta_{m,n-2} - 3(n+1)^{3/2} \delta_{m,n+2} + \sqrt{(n+1)(n+2)(n+3)(n+4)} \delta_{m,n+4} \right], \hspace{1cm} (B5)$$

$$\langle m|\hat{y}^6|n\rangle = \frac{1}{\sqrt{2}} \left[ -\sqrt{(n-3)(n-2)(n-1)n} \delta_{m,n-4} + n^{3/2} \delta_{m,n-2} - (n+1)^{3/2} \delta_{m,n+2} + \sqrt{(n+1)(n+2)(n+3)(n+4)} \delta_{m,n+4} \right], \hspace{1cm} (B6)$$

$$\langle m|\hat{y}^7|n\rangle = \frac{1}{2\sqrt{2}} \left[ \sqrt{(n-4)(n-3)(n-2)(n-1)n} \delta_{m,n-5} + 3n^{3/2} \delta_{m,n-3} - 3(n+1)^{3/2} \delta_{m,n+3} + \sqrt{(n+1)(n+2)(n+3)(n+4)(n+5)} \delta_{m,n+5} \right], \hspace{1cm} (B7)$$

$$\langle m|\hat{y}^8|n\rangle = \frac{1}{\sqrt{2}} \left[ -\sqrt{(n-4)(n-3)(n-2)(n-1)n} \delta_{m,n-5} + n^{3/2} \delta_{m,n-3} - (n+1)^{3/2} \delta_{m,n+3} + \sqrt{(n+1)(n+2)(n+3)(n+4)(n+5)} \delta_{m,n+5} \right], \hspace{1cm} (B8)$$

Next, we will consider the long-wavelength expansion of the integrals $B^{(1)}_{mn}$, $B^{(2)}_{mn}$, $B^{(3)}_{mn}$. These can be efficiently evaluated by using the Feynman parametrization, similar to what was done in Appendix A. We then have for $B^{(1)}_{mn}$

$$B^{(1)}_{mn}(\tilde{k}_x) = \frac{3h^2}{4} \int_{-1}^1 dw \int_0^1 dy \left[ (\tilde{k}_x/2)^2 (1 - w^2 y^2) + h(m-n) w y + \mu(m+n)/2 \right]^{5/2}.$$  \hspace{1cm} (B9)

Expanding for small $\tilde{k}_x$, we find

$$B^{(1)}_{mn}(\tilde{k}_x) = \frac{1}{(m-n)^2} \left[ \frac{1}{\mu_m^{1/2}} + \frac{1}{\mu_n^{1/2}} - \frac{2}{\mu_{(m+n)/2}} \right] + \left\{ \frac{1}{(m-n)^2} \mu_{(m+n)/2}^{3/2} + \frac{4}{(m-n)^2} \frac{1}{\mu_{(m+n)/2}^{1/2}} \right\} \left( \tilde{k}_x/2 \right)^2 + O(\tilde{k}_x^4).$$  \hspace{1cm} (B10)
\[ B_{mn}^{(2)}(\kappa_z) = \int_{-1}^{1} dw \int_{0}^{1} dy \frac{y}{\left[ (\kappa_z/2)^2(1 - w^2y^2) + h(m - n)wy + \mu_{(m+n)/2} \right]^{3/2}} 
+ 3(\kappa_z/2)^2 \int_{-1}^{1} dw \int_{0}^{1} dy \frac{w^2y^3}{\left[ (\kappa_z/2)^2(1 - w^2y^2) + h(m - n)wy + \mu_{(m+n)/2} \right]^{5/2}} 
= -\frac{4}{h^2(m - n)^2} \left[ \mu_m^{1/2} + \mu_n^{1/2} - 2\mu_{(m+n)/2}^{1/2} \right] - \frac{4}{h^2(m - n)^2} \left[ \frac{1}{\mu_m^{1/2}} + \frac{1}{\mu_n^{1/2}} \right] 
+ \frac{1}{\mu_{(m+n)/2}} + \frac{12\mu_{(m+n)/2}^{3/2}}{h^2(m - n)^3} \left( \mu_m^{1/2} + \mu_n^{1/2} - 2\mu_{(m+n)/2}^{1/2} \right) \left( \kappa_z/2 \right)^2 + O(\kappa_z^4). \tag{B11} \]

If \( m = n \), then this becomes

\[ B_{nn}^{(2)}(\kappa_z) = \frac{1}{\mu_m^{3/2}} - \frac{3}{4\mu_m^{5/2}} \left( \kappa_z/2 \right)^2 + O(\kappa_z^4). \tag{B12} \]

Finally, for \( B_{mn}^{(3)} \) we have

\[ B_{mn}^{(3)}(\kappa_z) = -\frac{3h^2}{4} \left( \frac{\kappa_z}{2} \right) \int_{-1}^{1} dw \int_{0}^{1} dy \frac{wy^2}{\left[ (\kappa_z/2)^2(1 - w^2y^2) + h(m - n)wy + \mu_{(m+n)/2} \right]^{5/2}} 
= \left[ \frac{1}{(m - n)^2} \left( \frac{1}{\mu_m^{1/2}} - \frac{1}{\mu_n^{1/2}} \right) \right] + \frac{4}{h(m - n)^3} \left( \mu_m^{1/2} + \mu_n^{1/2} - 2\mu_{(m+n)/2}^{1/2} \right) \left( \kappa_z/2 \right)^2 + O(\kappa_z^4). \tag{B13} \]

Note that this last integral is odd in \((m, n)\), in contrast to the first two integrals, and vanishes when \( m = n \).

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4. In the Meissner phases at zero field, this behavior is essentially due to the Goldstone mode. See the work by H. Schmidt in Z. Phys. 232, 443 (1970). One can recover \( \sigma \sim 1/k^2 \) simply by setting \( \omega = 0 \) in Schmidt’s result.
6. As in most of the literature to date [e.g., P. Nozières and W. F. Vinen, Philos. Mag. 14, 667 (1966)], we take the coefficient of this term to be 1.