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Hidden conservation law for sandpile models

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We point out that in certain classes of sandpile models, there exist conserved quantities, modulo the largest period of the system, under the toppling rule. In the deterministic case, explicit construction of such conserved quantities allows the exact solution of the models. With random seeding, the conservation laws also provide nontrivial information about the system.

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Sandpile models are considered to be one of the simplest models that may exhibit self-organized criticality [1]. In one spatial dimension, Kadanoff, Nagel, Wu, and Zhou [2] had studied a large class of interesting models. In particular, we shall consider the so-called "nonlocal models" in which N grains of sand at site i fall over N nearest sites to the right when the height difference at site i and site $i+1$ exceeds N. In a previous work, we have shown that the model with $N=2$ can be solved exactly when sand is dropped only along the left edge [3]. In this case, there is no self-organized criticality. The system of size L approaches a limit cycle consisting of N^L states. When a grain of sand is dropped, the slopes at C consecutive sites may be affected. The chance of finding a large C decreases exponentially as C increases. Roughly speaking, in order for the sand drop so as to affect the slopes of C consecutive sites, the configuration of these C sites has to be chosen in a specific way. The probability for this to happen is proportional to $N^{L-C}/N^L \sim N^{-C}$.

The purpose of this Brief Report is threefold. First, we shall give a complete characterization of the "allowed states" for the nonlocal models with random seeding. Second, we shall point out the existence of a conserved quantity, modulo N^L, under the toppling rule. Finally, we indicate how the conservation law allows us to solve the model for general N exactly when sand is dropped only along the left edge. When sand is dropped randomly over the lattice, the conservation law still provides useful information.

We consider a one-dimensional lattice of L sites labeled by $i=0,1,\ldots,L-1$. The height of the ith site is h_i and the boundary condition is $h_i=0$ for $i\geq L$. A given state is characterized by the vector $\langle h_0,h_1,\ldots,h_{L-1}\rangle$. When sand is dropped randomly over all sites, the system will reach a steady state in which only certain specific "allowed states" may appear. These states are characterized by two conditions [4]:

(a) $N(L-i)\geq h_i\geq(L-i)$ for all i,
(b) $\sigma_i\equiv h_i-h_{i+1}\leq N$ for all i.

The total number $\mathcal{N}_L$ of allowed states is

$$\mathcal{N}_L=\frac{1}{(L+1)}\binom{N(L+1)}{L}=\frac{[N(L+1)]!}{(L+1)!(NL+N-L)!} \tag{1}$$

and $\mathcal{N}_L/N^L\sim[N/(N-1)]^{(N-1)L}$ for large L. In the deterministic case, when sand is dropped only along the left edge, the "allowed states" need to satisfy two more conditions [3]:

(c) $\sigma_i\geq 0$ for all i,
(d) application of the "raising operator" I_i to any site $L-1\geq i\geq 0$ will trigger a sand slide,

where I_i operates on a state $\langle h_0,h_1,\ldots,h_{L-1}\rangle$ by $h_i\to h_i+N$, $h_{i+1}\to h_{i+1}+N-1$, $\ldots$, $h_{i+N-1}\to h_{i+N-1}+1$, while keeping the boundary condition $h_j=0$ for $j\geq L$ always. The two extra conditions reduce the number of allowed states to N^L.

The allowed states can be classified according to the conserved charge

$$Q=\sum_{k=0}^{L-1}b_{L-k}h_k, \tag{2}$$

where b_{L-k} are certain integers. Q is conserved modulo N^L under the toppling rule $h_i\to h_i-N$, $h_{i+1}\to h_{i+1}+1$, $\ldots$, $h_{i+N}\to h_{i+N}+1$, taking the boundary condition into account. In contrast, there are many quantities, such as the total sand number, which is conserved by the toppling rule only away from the boundary.

In general, there can be many solutions of b_k for Q. We shall be concerned with the ones satisfying the condition

$$b_L=1 \tag{3}$$

so that when sand is dropped at site $i=0$, the corresponding charge Q increases by 1. The existence of such a Q implies two things. First, Q takes on precisely N^L values. The total $\mathcal{N}_L$ "allowed states" are classified into N^L clusters according to their Q values. Second, Q specifies the "time development" of the system when grains of sand are dropped one by one along the left edge. Thus the steady-state behavior of the deterministic sandpile model is completely solved.

When $N=2$, the explicit form for Q is

$$Q=\sum_{k=0}^{L-1}(-2)^k h_k. \tag{4}$$

One notices that by dropping sand repeatedly at site k, a cycle of period 2^{L-k} will be excited. This is a generic feature for all N. We also note that Q is actually conserved under the operation of the raising operator I_i defined above.

For $N > 2$, no such simple form for Q in Eq. (4) exists. The coefficient b_{L-k} depends not only on k but also on L. To emphasize this dependence, we shall write b^L_{L-k} instead of b_{L-k}. It can be shown that

$$b^L_1 = N^{L-1}, \quad b^{L_1}_{L_1-k} = b^{L_2}_{L_2-k} \pmod{N^{L_1}} \quad \text{for} \quad L_1 < L_2. \tag{5}$$

Equation (5) implies $b^L_{L-k}/N^k = 1 \pmod N$. The construction of the explicit form for b^L_k is rather involved and will be described elsewhere. To get some feeling about the solution, we give an example for $N = 3$. In this case, b^L_{L-k} can be written as

$$b^L_{L-k} = 3^k \sum_{i=1}^{L-k} \tfrac{1}{2} 3^{i-1} \{ \tfrac{1}{2}[(-1+\sqrt{2}i)^{L-k-i} + (-1-\sqrt{2}i)^{L-k-i}] + 3^{L-k-i} \} l^L_i, \tag{6}$$

where l^L_i are certain integers. For $L = 11$, we may choose $l^{11}_1 = 1$, $l^{11}_6 = -1$, and $l^{11}_i = 0$ for $i \neq 1, 6$. For $L = 21$, the entries of l^{21}_i may be chosen to be

$$l^{21}_i = \langle 1,-1,0,0,1,0,-2,0,1,1,-1,1,-1,-2,1,1,-3,0,3,0,0 \rangle_i .$$

When sand is dropped randomly on all sites, the total of $\mathcal{N}_L$ "allowed states" is classified into N^L clusters according to their Q values. The dropping of one grain of sand at site k will induce a one-to-one mapping among the clusters with the change in Q, $\Delta Q = b_{L-k}$. This implies that the total probability of finding a state in a given cluster is N^{-L} for all clusters. The probability of finding a given state within a cluster, however, varies over a wide range.

We have pointed out the existence of a nontrivial conserved quantity in the one-dimensional nonlocal sandpile models. Existence of such conservation law in other sandpile models may very well be established along similar lines.

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