SCATTERING OF $p$-MODES BY SUNSPOTS. II. CALCULATIONS OF PHASE SHIFTS FROM A PHENOMENOLOGICAL MODEL

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ABSTRACT

We model the scattering of $p$-mode waves in a polytropic atmosphere by localized inhomogeneities in wave speed, pressure, and density of the medium. The effect of the inhomogeneities is attributed to a source term in the pressure wave equation. The inhomogeneous wave equation for the scattered waves is solved under the simplification of the Born approximation. From the solution for the scattered waves, we compute the phase shifts between the incoming and outgoing waves of individual modes.

We find that the variations of the computed phase shifts with degree $l$ and radial order $n$ of the modes show different behavior for inhomogeneities with different characteristic depths. Depths significantly shallower than the depth of the modes seem to show a phase shift dependence on $l$ and $n$ that is similar to the qualitative behavior of the observed phase shifts produced by sunspots. Direct quantitative comparison of the computed phase shifts with observations is limited to modes with lower degree ($l \leq 200$) in which the observed phase shifts are reasonably small so that the Born approximation is applicable. We find that for inhomogeneities with a wave speed contrast reasonable for sunspots, occupying a volume described by a characteristic depth $D \approx 10^8$ cm and horizontal radius $R \approx 2.5 \times 10^9$ cm, the computed phase shifts at lower $l$ range are in agreement with the observed phase shifts from sunspots in both their magnitudes as well as their variation with $l$ and frequency (or $n$).

Subject headings: scattering — Sun: oscillations — sunspots

1. INTRODUCTION

Observational studies by Braun, Duvall, & LaBonte (1988), Braun, LaBonte, & Duvall (1990), Braun et al. (1992), Braun (1995, henceforth Paper I), and Bogdan et al. (1993) have shown that sunspots scatter intermediate- and high-degree $p$-modes. It is found that sunspots absorb about 50% of the acoustic energy flux that is incident upon them and produce a shift in phase between the incoming and outgoing waves of individual modes. It is also found that sunspots scatter a measurable amount of incoming $p$-mode waves into outgoing modes of the same temporal frequency but different radial orders—a process termed mode mixing (Paper I). The wealth of observational information on the interaction between $p$-mode waves and sunspots offers the possibility that suitable models of scattering may yield important clues about the subsurface structure and evolution of solar activity.

A considerable amount of theoretical work has been done to explain the observed $p$-mode absorption by sunspots (as reviewed by Bogdan 1992). The emphasis of these efforts have been on understanding the physical mechanisms for absorption by magnetic flux tubes and explaining the observed magnitude of the absorption. There are yet no theoretical models which fully predict the observed magnitude and variation of absorption with degree $l$ and temporal frequency of the modes.

On the other hand, the presence of scattering phase shifts is readily expected because sunspots produce localized changes in the acoustic properties of the medium. For example, in Paper I we showed that the observed $p$-mode phase shifts can be caused by a local increase of wave speed in the volume occupied by a sunspot.

In this paper, in order to gain some insight from the observed $p$-mode phase shifts and their variation with mode degree $l$ and frequency, we take up the phenomenological approach suggested by Chou & Chen (1993) and Chou et al. (1995). The approach is similar to that of Brown (1990) in that the interaction between the sunspot and the acoustic waves is characterized by a source term which operates linearly on the local wave field and the scattered waves are computed using the Born approximation (BA). Rather than trying to solve for scattered waves in the presence of magnetic flux tubes, we model scattering by localized inhomogeneities in wave speed, pressure, and density in an adiabatically stratified polytropic atmosphere. These inhomogeneities reflect the local changes sunspots may introduce to the acoustic properties of the medium. We consider an isotropic increase of the effective wave speed inside the inhomogeneity. This can result from the change of compressibility of the medium due to the presence of magnetic pressure. However, the anisotropy caused by the magnetic tension is not considered here. We also assume that there is a decrease of gas pressure and density in the inhomogeneity, which is also expected inside a sunspot. These inhomogeneities in the static quantities of the medium are formulated.
into a source term in the pressure wave equation (similar to the procedure used by Rosenthal 1991, 1995). The inhomogeneous wave equation is then solved for the scattered waves under the simplification of the BA. We plan to study how the variations of the scattering phase shifts as a function of degree l, frequency ν, and azimuthal order m depend upon the characteristic spatial scales and contrast of the inhomogeneities. We do not attempt here to model the p-mode absorption seen in sunspots. Indeed, the use of the BA introduces a spurious energy emission of second order. Chou & Chen (1993) and Chou et al. (1995) discuss how absorption might be modeled phenomenologically by the use of a complex sound-speed inhomogeneity. We also note that our model does not predict any scattering of the fundamental (f) mode.

The rest of the paper will be structured as follows. In § 2 we will describe the formulation of the model and derive the scattering matrix from which we can obtain the p-mode phase shifts. In § 3 we will present and discuss the results of the computed phase shifts and compare them with the observed phase shifts produced by sunspots.

2. FORMULATING THE MODEL

The linearized equations for wave perturbations in a gravitationally stratified atmosphere are as follows:

$$\frac{\partial p'}{\partial t} + \nabla \cdot (p' \rho) = 0, \quad (1)$$

$$p' \frac{\partial u}{\partial t} + \nabla p' - \nu \rho' \mathbf{g} = 0, \quad (2)$$

$$\left( \frac{\partial p'}{\partial t} + u \cdot \nabla p' \right) - c^2 \left( \frac{\partial p'}{\partial t} + u \cdot \nabla \rho' \right) = 0, \quad (3)$$

where \(p', \rho', \) and \(u\) denote the wave perturbations in pressure, density, and velocity, \(p, \rho, \) and \(c\) represent the static pressure, density, and wave speed of the medium, and \(g\) is the gravitational acceleration. We now rewrite the static quantities (\(p, \rho, c^2\)) in the above equations as sums of their background values and the changes due to the inhomogeneity:

$$c^2 = c_0^2(z) + c_r^2(r, z), \quad (4)$$

$$p = p_0(z) + p_r(r, z), \quad (5)$$

$$\rho = \rho_0(z) + \rho_r(r, z). \quad (6)$$

In this paper we use the cylindrical coordinates \((r, \phi, z)\), where \(r\) is the horizontal radius, \(z\) is the vertical depth, and \(\phi\) is the azimuthal angle, and in the above we have assumed that the spatial variation of the inhomogeneity is axisymmetric. For the background values (\(c_0^2, p_0, \rho_0\)) of the static quantities, we assume an adiabatically stratified polytropic atmosphere, so that \(g = gz^\gamma\) is uniform, \(c_0^2 = gz^\gamma, p_0 \propto z^{\gamma+1}\), and \(\rho_0 \propto z^\gamma\), where \(\gamma\) is the polytropic index that relates to the ratio of specific heats \(\gamma\) via \(\gamma = 1 + 1/\alpha\). In this calculation we have chosen \(\alpha = 2.1\), which roughly approximates the conditions in the upper convection zone of the Sun. The deviations in the inhomogeneity from the background conditions are represented by \(c_r^2, p_r, \) and \(\rho_r\).

Substituting expressions (4), (5), and (6) into equations (1), (2), and (3), taking \(c_{rot}\) for the time dependence of \(p', \rho', \) and \(u\), and arranging all terms involving \(c_r^2, p_r, \rho_r\) on the right-hand side, we obtain the following:

$$i \omega p' + (u \cdot \nabla) p_0 + \rho_0 (\nabla \cdot u) = -(u \cdot \nabla) p_r - \rho_r (\nabla \cdot u), \quad (7)$$

$$i \omega p' + (u \cdot \nabla) p_0 + \rho_0 (\nabla \cdot u) = -(u \cdot \nabla) p_r - \rho_r (\nabla \cdot u), \quad (8)$$

$$\frac{d}{dt} p' + \omega \mathbf{r} \cdot \mathbf{g} = -i \omega p', \quad (9)$$

In the above we have ignored a higher order term \(O(c_r^2)\), retaining only terms \(O(c_r^2), O(p_r), \) and \(O(\rho_r)\) on the right-hand sides. Without the inhomogeneity, i.e., if \(c_r^2, p_r, \) and \(\rho_r\) are all set to zero, the above equations give the regular wave solutions for a plane-parallel polytropic atmosphere. We further eliminate \(\rho'\) and \(u\) in the left-hand sides of the above equations and obtain the following inhomogeneous wave equation for \(p'\):

$$\tilde{L} p' + \frac{\omega^2}{\tilde{z} + 1} p' = S, \quad (10)$$

where

$$\tilde{L} = \frac{g}{\alpha z^\gamma} \left( \frac{\partial}{\partial z} + \frac{\alpha}{z} \tilde{F} \right), \quad (11)$$

$$S = \frac{g}{\alpha z^\gamma} \left( \frac{\partial}{\partial z} + \frac{g \partial F}{\omega} \right), \quad (12)$$

$$F = \frac{1}{c_0^2} (u \cdot \nabla)p_1 - (u \cdot \nabla)p_0 + \rho_0^2 (c_r^2) (\nabla \cdot u). \quad (13)$$

In equation (10), \(\tilde{L}\) is a linear operator given by equation (11), where the factor \((g/\alpha z^\gamma)\) is introduced to turn the \(z\)-dependent part of \(\tilde{L}\) into a Sturm-Liouville—type operator. The effects of the inhomogeneity are attributed to a source term \(S\), which is expressed explicitly by equations (12) and (13). The source term \(S\) depends on the spatial distribution of \(c_r^2, p_r, \) and \(\rho_r\), and it also depends on the wave field itself through \(u, \) as in equation (13).

Solving equation (10) exactly is generally difficult. To make the problem tractable we employ the BA, under which we replace the wave field in the source term by the solutions \((p_{\rho}^{00}, \rho_{\rho}^{00}, u_{\rho}^{00})\) of the wave equations (eqs. [7], [8], and [9]) in the absence of the inhomogeneity. Therefore, \(S\) in equation (10) is approximated by

$$S^{00} = \frac{g}{\alpha z^\gamma} \left( \frac{\partial}{\partial z} + \frac{g \partial F^{00}}{\omega} \right), \quad (14)$$

$$F^{00} = \frac{1}{c_0^2} (u^{00} \cdot \nabla)p_1 - (u^{00} \cdot \nabla)p_0 + \rho_0^2 \frac{c_r^2}{c_0^2} (\nabla \cdot u^{00}). \quad (15)$$

It should be noted here that a more general form of the source term \(S^{00}\) for the pressure wave equation has been derived by Rosenthal (1995) that includes not only the effect of changes in wave speed, pressure, and density, but also describes the effect of the Lorentz force from magnetic fields. The source term we obtain here is consistent with his result, with the term representing the Lorentz force excluded.

The general solutions for wave perturbations \(p^{00}, \rho^{00}, \) and \(u^{00}\) in a plane-parallel, adiabatically stratified polytropic atmosphere are known. The pressure perturbation \(p^{00}\) satisfies the homogeneous wave equation:

$$\tilde{L} p^{00} + \frac{\omega^2}{\tilde{z} + 1} p^{00} = 0. \quad (16)$$

The solutions to equation (16) under the physical boundary conditions form a set of orthogonal eigenfunctions:

$$\psi_{s, k, m} = \Phi_{s, k}(z) J_m(kr) e^{im\phi}, \quad (17)$$

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where \( k > 0 \) is the horizontal wavenumber, \( m \) is the azimuthal order, and \( \Phi_{n,k}(z) \) are the vertical wave functions that satisfy the differential equation
\[
\frac{g}{\alpha} \frac{d^2 \Phi}{dz^2} + \frac{\alpha}{z} \frac{d \Phi}{dz} + \left( \frac{\alpha}{g \omega^2 z^2} - k^2 \right) \Phi = 0 .
\] (18)

The above equation is of Sturm-Liouville-type, and the solutions are
\[
\Phi_{n,k}(z) = \sqrt{\Gamma(\alpha + n)} \frac{1}{\Gamma(\alpha)} (2k)^{\alpha/2} z^{\alpha - 1} e^{-kz} M(-n, \alpha, 2kz), \quad k > 0 ,
\] (19)
where \( M(-n, \alpha, 2kz) \) is the Kummer function (see Abramowitz & Stegun 1965). The corresponding eigenfrequencies for the modes are as follows:
\[
\omega_{n,k}^2 = g k \left( \frac{\alpha + 2n}{\alpha} \right), \quad n = 0, 1, 2, 3, \ldots, k > 0 .
\] (20)

In the above, \( n = 0 \) corresponds to the f-mode, and \( n = 1, 2, 3 \ldots \) represent the p-modes of order \( n \). Figure 1 shows the \( k-\omega \) diagram resulting from the above dispersion relation, in which the polytropic index \( \alpha = 2.1 \) and the gravitational acceleration \( g = 2.8 \times 10^4 \) cm s\(^{-2}\). At a fixed value of \( k \), the set of eigenfunctions \( \Phi_{n,k} \) corresponding to different eigenfrequencies \( \omega_{n,k} \), \( n = 0, 1, 2, \ldots \), (as illustrated by the triangles in Figure 1) has the following orthogonal relation:
\[
\int_0^\infty \Phi_{n,k}(z) \Phi_{n',k}(z) \frac{1}{z^{\alpha + 1}} dz = \delta_{nn'} .
\] (21)

For \( n' = n \), the above integral corresponds to the depth integration of mode energy. Hence, the vertical wave function \( \Phi_{n,k} \) is energy normalized.

With \( p^{(0)} \) known, \( u^{(0)} \) and \( \mathbf{V} \cdot u^{(0)} \) in the source term (eq. [15]) can be obtained through the following relations:
\[
u^{(0)} = \frac{1}{i \alpha \rho \omega} \left( \frac{g^{(0)}}{c_0^2} \mathbf{g} - \mathbf{V} p^{(0)} \right) ,
\] (22)
\[
\mathbf{V} \cdot u^{(0)} = \frac{1}{i \alpha \rho \omega} \left[ \frac{\omega^2}{c_0^2} \left( \frac{g}{c_0^2} p^{(0)} - \frac{\partial p^{(0)}}{\partial z} \right) \right] .
\] (23)

Thus, with the simplification of the BA, the source term \( S^{(0)} \) is known and the inhomogeneous wave equation (10) has the following solution:
\[
p' = p^{(0)} + p^{(1)} ,
\] (24)
where \( p^{(0)} \) represents the solution of the corresponding homogeneous equation (16) at fixed frequency \( \omega \), and
\[
p^{(1)} = \int_0^{2\pi} d \phi \int_0^\infty \int_0^\infty d r_0 d z_0 [G(r, \phi, z, r_0, \phi_0, z_0)] S^{(0)}
\] (25)
gives the scattered waves due to the presence of the inhomogeneity. In the above equation, \( G(r, \phi, z, r_0, \phi_0, z_0) \) is the Green's function satisfying the following equation:
\[
\hat{L} G + \frac{\omega^2}{z^2 + 1} G = \frac{1}{r} \delta(r - r_0) \delta(\phi - \phi_0) \delta(z - z_0) ,
\] (26)
which describes waves from a point source at \((r_0, \phi_0, z_0)\) in the polytropic atmosphere.

To solve equation (26) we follow Rosenthal (1991, 1995) (see also Chou et al. 1995, which derives a more general form of Green's function in the case of a solar model). We first express the \( r \) and \( \phi \) dependence of \( G \) in terms of its azimuthal and

![Figure 1](https://example.com/image.png)

**Fig. 1.—** The \( k-\omega \) diagram for p-modes in a adiabatically stratified polytropic atmosphere. Here \( k \) denotes the horizontal wavenumber, and \( \omega_{n,k} \) denotes the angular frequency of the modes. The polytropic index used is \( \alpha = 2.1 \), and the gravitational acceleration is \( g = 2.8 \times 10^4 \) cm s\(^{-2}\).

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Fourier-Bessel components $g_{mk}$:

$$G = \sum_m \left[ \int_0^\infty g_{mk}(z) J_m(kr) k \, dk \right] e^{i\omega \phi},$$

(27)

where $g_{mk}(z)$ is the result of applying the azimuthal transform and then the appropriate Hankel transform to $G$:

$$g_{mk}(z) = \int_0^\infty \left[ \frac{1}{2\pi} \int_0^{2\pi} Ge^{-i\omega \phi} \, d\phi \right] J_m(kr) r \, dr.$$  

(28)

Substituting expression (27) into equation (26) and using the orthogonal properties of $e^{i\omega \phi}$ and $J_m(kr)$, we obtain the following:

$$\frac{g_1}{\alpha z^2} \left[ \frac{d^2 g_{mk}(z)}{dz^2} - \frac{\alpha}{z} \frac{dg_{mk}(z)}{dz} + \left( \frac{\alpha}{gz} w^2 + \frac{\alpha}{z^2} - k^2 \right) g_{mk}(z) \right]$$

$$= \frac{1}{2\pi} \delta(z - z_0) J_m(kr_0) e^{-i\omega \phi_0}.$$  

(29)

The corresponding homogeneous equation for the above equation is equation (18). At fixed frequency $\omega$, equation (18) under the physical boundary conditions has eigensolutions $\Phi_{n,k_n}(z)$ corresponding to eigenvalues $k_n$, where

$$k_n = \frac{\omega^2}{g} \frac{\alpha}{\alpha + 2n}, \quad n = 0, 1, 2, \ldots,$$

(30)

as illustrated by the circles in Figure 1, and $\Phi_{n,k_n}(z)$ are given by equation (19), in which $k$ is replaced by $k_n$. The set of functions $\Phi_{n,k_n}(z)$ corresponding to $k_n$, $n = 0, 1, 2, \ldots$ at fixed frequency $\omega$ have the following orthogonal property:

$$\int_0^\infty \Phi_{n,k_n}(z) \Phi_{n',k_n}(z) \frac{1}{z^2} \, dz = \frac{1}{2k_n} \delta_{n'n}.$$  

(31)

Note the above integral does not equal unity when $n = n'$ since each $\Phi_{n,k_n}(z)$ has been normalized according to equation (21), which uses a different weighting function. We can then expand $g_{mk}(z)$ in equation (29) as a sum of the eigenfunctions $\sum_n c_n \Phi_{n,k_n}(z)$ and determine the coefficients $c_n$ using the orthogonal relation (31). We obtain

$$c_n = \frac{1}{\pi \omega^2} \frac{\alpha}{k_n^2 - k^2} J_m(kr_0) e^{-i\omega \phi_0},$$

(32)

and therefore,

$$G = \sum_n \frac{1}{\pi \omega^2} \Phi_{n,k_n}(z_0) \Phi_{n,k_n}(z) e^{i\omega \phi} \int_0^\infty J_m(kr_0) J_m(kr) k \, dk.$$  

(33)

The integral in the above expression,

$$I = \int_0^\infty \frac{J_m(kr_0) J_m(kr)}{k_n^2 - k^2} k \, dk$$

(34)

is evaluated by contour integration (Rosenthal 1995), and the result corresponding to outward traveling waves is as follows:

$$I = \frac{in}{2} J_m(kr_0) H_m^{(2)}(k_n r), \quad r > r_0,$$

(35)

since we have chosen the time dependence to be $e^{i\omega t}$ (consistent with the definition in Paper I). Substituting the result of the integral into equation (33) yields the solution for Green's function:

$$G(r, \phi, z, r_0, \phi_0, z_0) = \frac{i}{2} \sum_n \frac{k_n^2}{\omega^2} \times \Phi_{n,k_n}(z_0) \Phi_{n,k_n}(z) J_m(k_n r_0) H_m^{(2)}(k_n r) e^{i\omega \phi - \phi_0},$$

(36)

which is a superposition of outgoing modes with different wavenumbers $k_n$ at fixed frequency $\omega$ (as illustrated by the circles in Fig. 1).

Having obtained the solution of the Green's function, one can compute the integral of equation (25) once $S^{(0)}$ is specified. The source term $S^{(0)}$ is given explicitly through equations (14), (15), (22), and (23), with the spatial coordinates $(r, z)$ replaced by $(r_0, z_0)$. We can see that $S^{(0)}$ operates linearly on $p^{(0)}$ and it is also a function of position $(r_0, z_0)$ depending upon the spatial distribution of $(c_r^2, p_r, \rho_r)$. Therefore, we can represent the complicated explicit form of the source term schematically as

$$S^{(0)} = S^{(0)}(p^{(0)}, r_0, z_0).$$

(37)

Thus, given an "initial" oscillation field (in the absence of inhomogeneities) consisting of a superposition of modes $(n', m')$ at frequency $\omega$,

$$p^{(0)} = \sum_{m'n'} a_{n,n',m'} \Phi_{n,n',m'}(z_0) J_m(k_n r_0) e^{i\omega \phi},$$

(38)

the source term $S^{(0)}$, because of its linearity with $p^{(0)}$ and the axisymmetry of the inhomogeneity, can be written as

$$S^{(0)}(p^{(0)}, r_0, z_0) = \sum_{m'n'} a_{n,n',m'} \frac{s_{n,n',m'}}{s_{n',n,m'}} \delta_{n'n} e^{i\omega \phi},$$

(39)

where $s_{n,n',m'}$ is equal to $S^{(0)}$ in which $p^{(0)}$ is replaced by its eigenfunction for mode $(\omega, n', m')$ in the $r_0$-$z_0$ plane.

$$s_{n,n',m'} = S^{(0)}[\Phi_{n,n',m'}(z_0) J_m(k_n r_0), r_0, z_0].$$

(40)

Substituting equations (36) and (39) into equation (25), we obtain the following for the scattered waves:

$$p^{(1)} = \sum_{m'n'} \left[ \sum_n \frac{s_{n,n',m'}}{s_{n',n,m'}} \mathcal{T}^{n,n',m'}(\omega) \right] \Phi_{n,n',m'}(z_0) J_m(k_n r) e^{i\omega \phi},$$

(41)

where

$$\mathcal{T}^{n,n',m'}(\omega) = \delta_{nn'} \frac{\alpha}{\omega^2} \int_0^\infty \frac{dz_0}{r_0} \int_0^\infty \frac{k_0^2}{\omega^2} \Phi_{n,n',m'}(z_0) \Phi_{n,n',m'}(z) J_m(k_n r_0) J_m(k_n r) e^{i\omega \phi}.$$  

(42)

In the above, $\mathcal{T}^{n,n',m'}(\omega)$ is the $T$ matrix which relates the amplitudes of the scattered modes with the amplitudes of the modes in the initial oscillation field. Because of the assumed axisymmetry of the inhomogeneity, the $T$ matrix is only nonzero when $m = m'$. On the other hand, $\mathcal{T}^{n,n',m'}(\omega)$ is generally nonzero for $n \neq n'$ and describes the coupling of modes between ridges of different $n$ values within the same unit frequency bin centered on $\omega$. The resulting total wave field ($p' = p^{(0)} + p^{(1)}$) is therefore

$$p' = \sum_m \sum_{n'n} a_{n,n',m} \Phi_{n,n',m}(z_0) J_m(k_n r) e^{i\omega \phi}$$

$$+ \sum_m \sum_{n'n} \left[ \sum_{n''} \frac{s_{n,n',m''}}{s_{n'',n,m''}} \mathcal{T}^{n,n',m''}(\omega) \right] \Phi_{n,n',m''}(z_0) J_m(k_n r) e^{i\omega \phi}.$$  

(43)
In the above we have without the loss of generality replaced all the primed indices \((n', m')\) in \(p'(0)\) with unprimed indices \((n, m)\).

We may regroup the total wave field \(p'\) into incoming and outgoing waves as follows:

\[
p' = \sum_{m} \sum_{n} \frac{1}{2} a_{n,n',m} \Phi_{n,n',m}(z) H_{0}^{(1)}(k_{r} r) e^{i m \phi} + \sum_{m} \sum_{n} \sum_{n'} \frac{1}{2} a_{m,m',n} \left[ \delta_{m',m} \delta_{n,n'} + 2 F_{m}^{n}(\omega) \right] \times \Phi_{n,n',m}(z) H_{0}^{(1)}(k_{r} r) e^{i m \phi},
\]

from which we identify the scattering matrix,

\[
F_{m}^{n}(\omega) = \delta_{m',m} \delta_{n,n'} + 2 F_{m}^{n}(\omega).
\]

The scattering matrix relates the amplitude of the incoming modes with those of the outgoing modes. With the scattering matrix (or \(S\) matrix) known, one can compute a phase shift \(\delta\) defined as the argument of the diagonal component of the \(S\) matrix, i.e.,

\[
\tan \delta = \frac{\text{Im} \left[ F_{m}^{n}(\omega) \right]}{\text{Re} \left[ F_{m}^{n}(\omega) \right]}.
\]

This phase shift is equivalent to the observed phase shift between the incoming and outgoing waves of mode \((\omega, n, m)\) as measured in Paper I (see the discussion in the Appendix of Paper I).

We need to specify the spatial distribution of the inhomogeneity \((c_{1}^{2}, \rho_{1}, \rho)\) in order to compute the \(T\) and \(S\) matrices derived above. So far, little is known about the subsurface structure of sunspot flux tubes. For this calculation, as discussed in the Introduction, we take a phenomenological approach to model the changes of the acoustic properties inside the sunspot. We assumed for the inhomogeneity an increase in the square of the effective wave speed \(c_{1}^{2}\) in equation (3) to reflect the decrease of the compressibility of the medium due to the presence of the magnetic pressure. In this way, we have relaxed the strict definition of \(c_{1}^{2}\) as the square of the adiabatic sound speed of an ideal gas so as to include the dynamical effect due to the addition of the magnetic field. Our model, however, does not consider the effect of anisotropy that can be introduced by the magnetic tension force. We define the "contrast" of the wave speed to be the ratio \((c_{1}^{2} / c_{0}^{2})\) between the change due to the inhomogeneity and the background value. In our present calculations we assume that the contrast is described by a Gaussian distribution in both the vertical and horizontal directions, i.e.,

\[
\frac{c_{1}^{2}}{c_{0}^{2}} = \epsilon \exp \left( -\frac{x^{2}}{D^{2}} \right) \exp \left( -\frac{R^{2}}{R_{0}^{2}} \right).
\]

Figure 2a shows the contours of constant contrast \((c_{1}^{2} / c_{0}^{2})\) in the \(r-z\) plane. Here, \(\epsilon\) is a positive constant representing the peak contrast at the surface point \(P\), and \(D\) and \(R\) are the characteristic depth and horizontal radius of the inhomogeneity. If the increase of the effective wave speed is due to the presence of magnetic pressure, then there should be a corresponding decrease in gas pressure. To specify this decrease, we estimate that the change in the square of the effective wave speed \(c_{1}^{2}\) is equal to the square of the Alfven speed. Thus, the magnetic pressure is approximately equal to \((1/2)\rho_{0} c_{A}^{2}\). To maintain pressure balance the change in gas pressure is then simply

\[
\rho_{1} = -\frac{1}{2} \rho_{0} c_{A}^{2}.
\]

There should also be a corresponding density decrease, as is expected within sunspots. Given the change in the gas pressure, the density change can be estimated by considering vertical hydrostatic equilibrium (ignoring possible static forces from the magnetic field in the vertical direction). Thus,

\[
\rho_{1} = \frac{1}{g} \frac{\partial p_{1}}{\partial z}.
\]

The resulting depth variations of \((c_{1}^{2} / c_{0}^{2})\), \(p_{1}/p_{0}\), and \(\rho_{1}/\rho_{0}\) along the center axis at \(r = 0\) are shown in Figure 2b. We then substitute the specified \((c_{1}^{2}, \rho_{1}, \rho_{0})\) into equation (15) to obtain the explicit form of \(S_{0}^{(0)}\) and \(S_{0}^{(0)} m, m'\) through equations (14), (15), (22), (23), and (40). The resulting source function \(s_{0}^{(0)} m, m'\) is found to be directly proportional to \(\epsilon\), which is defined in equation (47) as the peak contrast of the inhomogeneity. With the explicit form of \(s_{0}^{(0)} m, m'\) known, one can numerically calcu-
late the integral in equation (42) to obtain $\mathcal{F}^m_{n,-m}(\omega)$. An advantage of choosing the spatial variation described in equation (47) for the inhomogeneity is that the radial and vertical parts of the integration in equation (42) can be carried out separately as one-dimensional integrals.

We note that under the BA the $T$ matrix is purely imaginary, and from equations (45) and (46),

$$
\tan \delta = -2i\mathcal{F}^m_{n,-m}(\omega) = \frac{2\pi}{\omega^2} \int_0^\infty dz_0 \int_0^\infty r_0 dr_0 [k^2 \Phi_{n,k}(z_0) \alpha_m(k, r_0) \phi_{n,m}(0, r_0)].
$$

This is the formula that we use in this paper to compute the phase shifts between the incoming and outgoing wave modes. Since we have not included any dissipative mechanism, there should be in principle no absorption of energy. However, the first-order BA produces a spurious energy emission $\eta$ of second order for each incoming mode $(\omega, n, m)$:

$$
\eta = \sum_{n,m} 4 |\mathcal{F}^m_{n,-m}(\omega)|^2.
$$

In this paper we focus on calculating the scattering phase shifts produced by the inhomogeneity and compare them with the observed $p$-mode phase shifts produced by sunspots. The results of our calculation are shown in the next section.

In general, the BA is accurate when the amplitudes of the scattered waves are small in comparison to the incoming waves, i.e., when $|\mathcal{F}^m_{n,-m}| \ll 1$. If $|\mathcal{F}^m_{n,-m}| \geq 1$, e.g., when the phase shifts $\delta \geq 45^\circ$, then the BA no longer gives a good approximation of the true solution. In the Appendix we present a comparison between the exact solution and the solution under the BA for scattering of acoustic waves by an infinite cylinder in a uniform atmosphere. This example helps to give us a practical sense of the range of validity of the BA.

3. RESULTS AND DISCUSSION

In this section we present the scattering phase shifts $\delta$ computed by numerically evaluating the integral in equation (50). First, we study the qualitative variation of $\tan \delta$ with the azimuthal order $m$, degree $l$, and radial order $n$ of the modes for inhomogeneities with different characteristic depth $D$ and horizontal radius $R$ defined in equation (47). The constant $\epsilon$, representing the peak contrast of the inhomogeneity, acts simply as a constant of proportionality in the calculation of $\tan \delta$, since $\phi_{n,m}(0)$ is directly proportional to $\epsilon$. Hence, for the purpose of studying the qualitative dependence on mode degrees $(m, l, n)$, we consider $\tan \delta$ divided by the constant $\epsilon$. When the phase shift $\delta$ is small, i.e., when the BA is valid, the behavior of $\tan \delta$ accurately represents the behavior of $\delta$.

It should be noted here that our scattering model is derived for a plane parallel atmosphere which is infinite in the horizontal directions. Thus, the allowed values for the horizontal wavenumber $k$ of the modes are continuous (i.e., the $p$-mode ridges are continuous curves). This represents a good approximation to the conditions of high $l$ solar $p$-modes. The appropriate relation for converting the horizontal wavenumber $k$ of our model to the degree $l$ that describes solar $p$-modes is as follows:

$$
k^2 = \frac{l(l+1)}{R_\odot^2},
$$

where $R_\odot$ is the solar radius. We further note that in this paper we focus on computing phase shifts for $p$-modes (with order $n \geq 1$). It can be shown that any localized inhomogeneities in wave speed, pressure, and density produce zero phase shifts for the $f$-mode ($n = 0$).

We first explore the functional behavior of our model phase shifts with azimuthal order. Our goal is to find a value of our radial parameter $R$ which gives qualitative agreement with the $m$ dependence of the observed phase shifts reported in Paper I. The top panel of Figure 3 shows the $m$ variation of $\tan \delta$ normalized to the corresponding value of $\tan \delta$ at $m = 0$, for a set of $l$ values (the results shown are essentially independent of the radial order $n$, as can be seen in eq. [53] below). Here we have only shown the result for positive $m$, since $\tan \delta$ is symmetric with respect to the axis of $m = 0$. It can be seen that $\tan \delta$ peaks at $m = 0$ and decreases with increasing $m$ in a Gaussian-like form, which is qualitatively consistent with the $m$ dependence of the observed phase shifts shown in Figure 12 of Paper I.

As was discussed in Paper I (see also Braun et al. 1988) the quantity $m/k$ is used to denote the "impact radius" of an incident $p$-mode, within which the $p$-mode energy falls to zero. For a scatterer of finite radius $R$ one expects that the phase shift should decrease to zero when the impact radius $m/k$ of the

Fig. 3.—The variation with $m$ of $\tan \delta$ normalized to the corresponding value of $\tan \delta$ at $m = 0$. The top panel shows $\tan \delta/(\tan \delta)_{m=0}$ as a function of $m$ for a set of different $l$ values and $n = 3$. The bottom panel is the same as the top one except that $\tan \delta/(\tan \delta)_{m=0}$ is plotted as a function of $m/k$. We can see that the different curves in the top panel have collapsed approximately into a single Gaussian curve that has a width of $m/k = R$ (at which $\tan \delta$ has reduced to $1/e$ of its value at $m = 0$).
mode becomes comparable to or larger than $R$. In the bottom panel of Figure 3 we plot the same values of phase shift shown in the top panel, but this time as a function of $m/k$. One observes that all of the curves corresponding to different $l$ (or $k$) now collapse approximately into a single Gaussian-like curve. We note that when the impact radius $m/k$ is equal to the radius $R$ of the scatterer, the phase shift has decreased from its value at $m = 0$ by roughly a factor of $1/e$.

For our specific choice of the spatial variation of the inhomogeneity described by equations (47), (48), and (49), the variations with $x$ and $r$ are decoupled. We find that, to a high degree of accuracy, $\tan \delta/\tan \delta_m$ can be expressed as follows:

$$\frac{\tan \delta}{\tan \delta_m} \approx \int_0^\infty \frac{d\delta}{\delta} \exp \left(-\frac{m^2}{R^2}J_0^2(k_0 r)\right) r_0 dr_0,$$

(53)

which is a function of $m$ and $k$ (or $l$) and is independent of $n$ and the depth $D$ of the inhomogeneity. The result that the curves of $\tan \delta/\tan \delta_m$ versus $m/k$ all collapse closely (but not exactly) into a Gaussian curve in the bottom panel of Figure 3 is a consequence of our special choice of the Gaussian distribution for the inhomogeneity. However, the result that all the curves should fall asymptotically to zero when $m/k > R$ is a general result for localized scatterers with a characteristic radius $R$.

As discussed in Paper I, the behavior of the observed phase shifts appears consistent with a nearly constant scattering radius. This is shown in Figure 4, where we have plotted in different panels the observed phase shifts $\delta$ as measured in sunspot NOAA 5254 as a function of $m/k$ for different $n$ and $l$ values. To the points in each panel we have also fitted a Gaussian function $A \exp \left[-(m/k)^2/R^2\right]$, where the fitted parameter $R$ shall indicate the characteristic radius of the scatterer as "seen" by the $p$-mode waves. The fitted values of $R$ (as shown in the panels for different $n$ and $l$ values) are close in magnitude, ranging between $2 \times 10^9$ cm and $3 \times 10^9$ cm, with a mean value of about $2.5 \times 10^9$ cm. We do not observe any systematic dependence of the value of $R$ with any mode property (such as the lower turning point of the mode). The fluctuations of the fit values of $R$ are consistent with the uncertainties of the fitting procedure given the observed errors. Hence, for the remainder of the paper we have chosen $R = 2.5 \times 10^9$ cm for our calculations.

For studying the $l$ and $n$ dependences of the computed phase shifts, we consider inhomogeneities with different characteristic depths $D$. We have chosen a set of $D$ values that range from depths significantly shallower than the acoustic cavities of the $p$-modes to depths that are comparable to or exceed the depths of the mode cavities. The lower turning point of each mode $z_l$ is defined as the depth at which the local total wavenumber $\omega/c_0$.

**Fig. 4.** The observed phase shifts $\delta$ from sunspot NOAA 5254, plotted as a function of $m/k$, for various $l$ and $n$ values. A Gaussian of the form $A \exp \left[-(m/k)^2/R^2\right]$ is fitted to the points in each panel, where the fitted parameter $R$ indicates the radial extent of the scatterer as "seen" by the $p$-modes.
is equal to the horizontal wavenumber $k$, and hence $z_i = (2\pi g)(\omega^2/k^2)$. Figure 5 shows $z_i$ for modes along three different ridges ($n = 2, 3, 4$) with $l = 0$ to 500. For comparison, the sets of $D$ values chosen are marked as dotted lines in Figure 5.

Figure 6 shows $\tan \alpha/\epsilon$ as a function of $l$ along individual $p$-mode ridges $n = 2, 3, 4$ for inhomogeneities with different characteristic depths $D$. In all cases here the azimuthal order is set to $m = 0$. As mentioned earlier, $\epsilon$ is simply a free constant which represents the peak contrast of the inhomogeneity at the surface. We find that the functional variation of $\tan \alpha$ with $l$ changes for different depths $D$ of the inhomogeneity. When $D$ is very large (as in the bottom two panels), being significantly greater than the lower turning point of high $l$ $p$-modes, $\tan \alpha$ increases approximately linearly with $l$. In this case the scattering phase shift is not sensitive to the depth of the modes but is roughly proportional to $kR$. As $D$ decreases and becomes shallower than the lower turning points of the modes, the increase of $\tan \alpha$ shows a higher order power-law dependence with $l$ than the linear dependence. The reason for this more sensitive increase with $l$ is that when the inhomogeneity becomes shallower than the depths of the modes, $\tan \alpha$ also depends upon the depth distribution of mode energy relative to the inhomogeneity. Increasing $l$ reduces the lower turning point of the modes so that a greater percentage of the mode energy is confined within the depth of the inhomogeneity and hence increases the phase shift. In comparison, the observed phase shifts shown in Figures 13 and 14 of Paper I also seem to have a faster than linear increase with $l$, similar to the case with smaller $D$ values.

The variation of $\tan \alpha$ with $n$ is more complicated and also depends on the choice of $D$. This is shown in Figure 7, in which we have plotted $\tan \alpha/\epsilon$ as a function of frequency $\nu$, where $2\pi \nu = gk(\alpha + 2n)/\alpha$, for fixed values of degree $l$ (or horizontal wavenumber $k$). The azimuthal order is set to $m = 0$ as before. We find that for shallow inhomogeneities, as shown in the top two panels, $\tan \alpha$ increases with increasing $\nu$, peaks at a certain frequency, and then gradually falls at higher $\nu$. The frequency at which $\tan \alpha$ reaches maximum appears to be insensitive to the values of $l$, e.g., for $D = 10^8$ cm, $\tan \alpha$ peaks at $\nu \sim 4$ mHz regardless of the value of $l$. We find that the qualitative behavior discussed above for shallow inhomogeneities is consistent with the observed frequency variation of phase shifts for fixed values of $l$, as shown in Figure 17 of Paper I. If $D$ is made to be comparable to the depths of the lower turning points of the modes (as in the middle two panels of Fig. 7), $\tan \alpha$ is found in general to decrease with increasing $\nu$. At even higher values of $D$ (see the bottom two panels) the decrease of the phase shifts with frequency becomes less pronounced. Eventually, when $D$ exceeds the lower turning points of all the modes, the value of $\tan \alpha/\epsilon$ becomes nearly constant for different $\nu$ (or $n$). Overall, by comparing with the observations, we find that the frequency dependence obtained from cases with $D \leq 10^8$ cm best resembles the qualitative variation of the observed phase shifts with frequency.

We have evaluated the relative contributions of individual terms in equation (15) to the magnitude of $\tan \alpha$. We find that the third term, which corresponds to the effect of $c_2^3$, is the dominant one and largely controls the variation of $\tan \alpha$ with $l$ and $\nu$. The contributions from the first and the second terms, corresponding respectively to the effects of $p_1$ and $p_2$, are in general significantly smaller than that of the $c_2^3$ term, except when the inhomogeneity becomes very shallow ($D < 10^8$ cm). In this case, the contribution of the $p_1$ term becomes comparable to that of the $c_2^3$ term, although both show a similar dependence on $l$ and $n$ of the modes. Hence, the qualitative variations of $\tan \alpha$ with $m$, $l$, and frequency $\nu$ (or $n$) presented above would remain unchanged if we only include the term corresponding to $c_2^3$ and ignore the effects of $p_1$ and $p_2$. 

*Fig. 5.—The depth of the lower turning points $z_i$ for modes along ridges $n = 2, 3, 4$. The dotted lines mark the set of depth values $D$ for the inhomogeneity chosen for our calculations.

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We have shown that studying the variation of $\tan \delta$ with $m$, $l$, and frequency $\nu$ of the modes can yield values of $D$ and $R$ that produce phase shifts that show similar qualitative behavior as the observed results reported in Paper I. We now seek to determine the value of peak contrast $\epsilon$ necessary to produce the magnitude of phase shifts seen in the observations. From Paper I we can see that the observed phase shifts produced by sunspots span a fairly large range, in some cases exceeding 90°, and are beyond the regime described by the BA. However, at lower degrees ($l \leq 200$) the observed phase shifts are reasonably small ($\delta < 45°$), and the BA is applicable. In order to reproduce the magnitude of the observed phase shifts at lower $l$ range (e.g., $\delta \sim 20°$ at $l = 200$, $n = 3$), we find that the required value of peak contrast is $\epsilon \approx 1$. A peak contrast of this magnitude in the change of wave speed in the inhomogeneity relative to the surrounding medium is reasonable based on the conditions inside sunspots. It is known that at the photospheric level the magnetic pressure produced by the field of a few kilogauss inside sunspots is approximately equal to the thermal pressure of the surrounding plasma. Therefore, one can expect a change of effective wave speed inside a sunspot $c_A^2 \sim c_S^2 \sim c_0^2$, where $c_A^2$ is the square of the Alfvén speed and $c_0^2$ as defined earlier is the square of the background sound speed.

Figure 8 shows a direct comparison of the observed phase shifts from a sunspot presented in Paper I (top) with the computed phase shifts $\delta$ obtained for an inhomogeneity with $\epsilon = 1$, $D = 10^6$ cm, and $R = 2.5 \times 10^9$ cm (bottom). The phase shifts $\delta$ are shown as a function of $l$ along three $p$-mode ridges ($n = 2$, 3, 4). For the computed phase shifts the azimuthal order is $m = 0$. As discussed in Paper I, the observed phase shifts have been averaged over an interval of azimuthal order $(-m_0$ to $+m_0)$ to reduce the noise; $m_0$ is the azimuthal order at which the phase shift is about one-half of the value at $m = 0$. Based on the $m$ dependence of phase shift shown earlier in this paper, the effect of this averaging is only to reduce the magnitude of the resulting $\delta$ by about 20% in comparison to $\delta$ at $m = 0$; it does not alter the variation of $\delta$ with $l$ or $n$. Hence, the true phase shifts at $m = 0$ produced by the sunspot would be slightly greater (by 20%) than the observed results shown here. With this in mind, we find that in the lower $l$ range ($l \leq 200$) the computed phase shifts from the inhomogeneity are consistent with the observed phase shifts from the sunspot in both the magnitude and the dependence on $l$ and $n$. In the higher $l$ range ($l \geq 250$), however, the BA is no longer a good approximation, and the computed results significantly underestimate the observed phase shifts. Figure 9 shows a direct comparison of the frequency ($\nu$) variation of phase shifts at fixed values of $l = 185$ and 225 between the observation (top two panels) and the computed results (bottom two panels). The parameters used for the inhomogeneity are the same as those in Figure 8, and the azimuthal order $m = 0$. For the observed results, the same kind of averaging over $m$ mentioned above is carried out to reduce the noise. We find that the computed phase shifts from the inhomogeneity show magnitudes and frequency variation.
similar to the observed phase shifts from sunspots. As shown in both the observed and the computed results, the phase shift $\delta$ increases with frequency $\nu$ until it reaches a maximum at $\nu \sim 4$ mHz, above which the phase shift $\delta$ levels and gradually falls.

In summary, we have described in this paper a model of $p$-mode scattering by localized inhomogeneities in wave speed, pressure, and density of the medium (as described in eqs. [47], [48], and [49]). These inhomogeneities represent an approximation to the changes sunspots may introduce in the local acoustic properties of the atmosphere. We find that some major features of the observed $p$-mode phase shifts, e.g., their $l$ and frequency (or $n$) dependences, may be reproduced by our model if the characteristic depth $D$ of the inhomogeneity is small (on the order of 1000 km). By comparing the $m$ dependence of the phase shifts with observations, we find that the inhomogeneity should have a characteristic radius $R$ of approximately 25,000 km, which is similar to the penumbra radius (approximately 20,000 km) of the observed sunspot. Furthermore, matching the magnitude of the observed phase shifts (at $l \leq 200$) requires an increase of the effective wave speed within the inhomogeneity to approximately $\sqrt{2}$ times the sound speed of the surrounding medium. The above results suggest that the change of the acoustic properties in sunspots compared to the surrounding plasma, or equivalently, the magnitude of the magnetic pressure in comparison to the external gas pressure, is significant within a short depth (approximately 1000 km) from the photosphere. This is in agreement with many existing theoretical pictures of sunspot flux tubes (see, e.g., Meyer et al. 1974; Meyer, Schmidt, & Weiss 1977).

In the present model we have assumed an isotropic increase of effective wave speed for the inhomogeneity. This reflects the change of compressibility due to the addition of magnetic pressure. However, the model has not taken into account the effect of magnetic tension, which introduces anisotropy into the medium. Our future work will incorporate the Lorentz force into the inhomogeneous equation for the scattered waves and hence extend our model to calculate $p$-mode scattering by realistic magnetic flux tubes in the solar convection zone.

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APPENDIX
SCATTERING BY AN INFINITE CYLINDER IN A UNIFORM ATMOSPHERE: A COMPARISON BETWEEN THE EXACT SOLUTIONS AND THE SOLUTIONS UNDER THE BORN APPROXIMATION

In this Appendix we consider the scattering of acoustic waves by an infinite cylinder of radius $a$, with axis along the $z$-direction, embedded in a uniform atmosphere of sound speed $c_s$ and density $\rho_s$. The material inside the cylinder has sound speed $c_p$ and density $\rho_p$, which may be different from those of the surrounding atmosphere. The scattering problem can be solved exactly by matching the solutions inside and outside the cylinder at the boundary, requiring the continuity of pressure and normal velocity perturbations.
SCATTERING OF \( p \)-MODES BY SUNSPOTS. II.

(see e.g., Morse & Ingard 1968). The resulting wave field outside the cylinder can be expressed as follows:

\[
p' = \sum_{m} \int_{-\infty}^{\infty} dk_x \int_{0}^{\infty} dk_{re} A_m(k_{re}, k_z) [J_m(k_{re} r) + \mathcal{F}_m(k_{re}, k_z) H_m^{(2)}(k_{re} r)] e^{i k_x x} e^{i \omega t} dt ,
\]

where

\[
\mathcal{F}_m(k_{re}, k_z) = \frac{\rho_1 k_{re} J_m(k_{ri} a) J_m(k_{ri} a) - \rho_2 k_{ri} J_m(k_{ri} a) J'_m(k_{ri} a)}{\rho_2 k_{ri} J'_m(k_{re} a) J_m(k_{ri} a) - \rho_1 k_{re} H_m^{(2)}(k_{re} a) J_m(k_{ri} a)} .
\]

In the above, \( J'_m(x) \), and \( H_m^{(2)}(x) \) represent the derivatives of the Bessel functions with respect to their argument \( x \), \( k_z \) is the vertical wavenumber which is the same inside and outside the cylinder, \( k_{ri} \) and \( k_{re} \) are, respectively, the horizontal wavenumber inside and outside the cylinder, \( \omega \) is the temporal frequency, and

\[
\frac{\omega^2}{c_l^2} = k_{ri}^2 + k_z^2 ,
\]

\[
\frac{\omega^2}{c_s^2} = k_{re}^2 + k_z^2 .
\]

The phase shift \( \delta \) between the incoming and outgoing waves of mode \((m, k_{re}, k_z)\) can be computed from the complex coefficient \( \mathcal{F}_m(k_{re}, k_z) \):

\[
\tan \delta = \frac{\text{Im} \left[ 1 + 2 \mathcal{F}_m(k_{re}, k_z) \right]}{\text{Re} \left[ 1 + 2 \mathcal{F}_m(k_{re}, k_z) \right]} .
\]

**Fig. 10.**—Scattering phase shifts \( \delta \) produced by an infinite cylinder in which the material has different sound speed and density compared to the surrounding uniform atmosphere. The solid curves represent the exact solutions, and the dashed curves the solutions under Born approximation. For all the cases shown here, the waves have \( m = 0 \) and \( k_z a = 0 \), i.e., normal incidence. The top panel shows examples with only sound-speed change in the cylinder, i.e., \( \epsilon_s = 0 \). The lower two curves correspond to \( \epsilon_s = 0.1 \), and the upper two curves correspond to \( \epsilon_s = 0.3 \). The middle panel shows a case with only density change in the cylinder. The bottom panel shows examples where there are both sound-speed and density changes. The lower two curves correspond to \( \epsilon_s = 0.1, \epsilon_s = 0.1 \), and the upper two curves correspond to \( \epsilon_s = 0.3 \) and \( \epsilon_s = 0.3 \).
We can also evaluate the above scattering problem under the BA. Taking similar steps described in the beginning of § 2, we can derive the inhomogeneous wave equation for pressure perturbation $p'$ for the current problem:

$$\nabla^2 p' + \frac{\omega^2}{c_e^2} p' = S^{(0)},$$

where

$$S^{(0)} = \frac{\omega^2}{c_e^2} \epsilon_e p^{(0)} + \nabla p^{(0)} \cdot \nabla \epsilon_p.$$

In the above, $\epsilon_e \equiv (c_1^2 - c_2^2)/c_2^2$, $\epsilon_e \equiv (\rho_1 - \rho_2)/\rho_1$, and $p^{(0)}$ denotes the wave solutions in the uniform atmosphere without the scatterer. For the source term $S^{(0)}$, we have ignored a higher order term $O(\epsilon_p \epsilon_e)$, retaining only terms $O(\epsilon_e)$ and $O(\epsilon_p)$, and we have employed the BA by replacing the wave field $p'$ with $p^{(0)}$. We then solve the corresponding Green's function and compute the scattered waves as described in § 2. The resulting solution for the wave field outside the cylinder can again be expressed as equation (54), in which the coefficient $\mathcal{T}_m(k_{re}, k_e)$ now becomes

$$\mathcal{T}_m(k_{re}, k_e) = \frac{i \pi}{2} \left\{ \frac{\omega^2}{c_e^2} \epsilon_e \int_0^a \left[ J_m(k_{re} r_0) \right]^2 r_0 dr_0 - \epsilon_e k_{re} a \mathcal{J}_m(k_{re} a) J_m(k_{re} a) \right\}.$$

From the above complex coefficient (which is pure imaginary according to BA) we can compute the phase shift through equation (58).

In Figure 10 we compare in a few examples the scattering phase shifts $\delta$ computed from the exact solution (eq. [55]) with those obtained under BA (eq. [61]). The solid lines represent the exact solutions, and the dashed lines represent the solutions under BA. For all the cases shown here, the waves have $m = 0$ and $k_e a = 0$, which correspond to normal incidence. Figure 10 (top) shows examples with only sound-speed change in the cylinder ($\epsilon_e = 0.1$ and 0.3, $\epsilon_p = 0$). We can see from the exact solutions that overall the phase shift increases linearly with $k_{re} a$, showing slight undulations with a period of $\pi$ radians in $k_{re} a$. The result under BA resembles the exact solution when the phase shifts are small (less than 45°). When the phase shift becomes greater than about 45°, the two solutions begin to diverge significantly, as indicated in the upper two curves. The middle panel shows a case where there is only a density difference ($\epsilon_e = 0$, $\epsilon_p = 0.3$) between the cylinder and its surroundings. The resulting phase shifts show very different variation with $k_{re} a$ compared to those produced by sound-speed changes. In this case the phase shifts oscillate between $\pm 20^\circ$, and the results under BA agree well with the exact solutions. The bottom panel shows examples where both the sound speed change and density change are present in the cylinder. Overall, we find from the comparison that the solutions under BA generally are a fairly good approximation of the exact solutions when the scattering phase shifts are below about 45°. Under BA a phase shift of 45° corresponds to the situation where the amplitude of the scattered mode is equal to the amplitude of the incident mode.

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