Diamagnetism in the dissipative regime

T. M. Hong
Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CB3 0HE, United Kingdom

J. M. Wheatley
Superconductivity Research Center, Cambridge University, Madingley Road, Cambridge CB3 0HE, United Kingdom

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The diamagnetic susceptibility $\chi_d$ of a two-dimensional charged-particle system in the dissipative regime $\hbar \tau^{-1} >> kT_0$ is $\chi_d = -(14/\pi) n\mu_B^2 \tau / \hbar$. (Here $\tau$ is the inelastic scattering lifetime, $T_0$ is the bare degeneracy temperature, $n$ is the number density, and $\mu_B$ is the Bohr magneton.) The renormalized $\chi_d$ due to an anomalous current relaxation parametrized by a high-frequency dissipative cutoff $\Omega_c$ is $\chi_d \sim \mu_B^2(\tau + 1/\pi\Omega_c)$. The Hall number is $n_H \sim n_H^0(1 + 1/\pi\Omega_c \tau)$. Possible significance of these results for cuprate superconductors in their normal states is pointed out.

There have been several attempts recently to account for the marked temperature dependence of the Hall coefficient of cuprate superconductors in the normal state\(^1\) within the framework of strongly correlated-single-band models.\(^2\) In the gauge theories\(^3\) the hole and spin species couple to a gauge field (fictitious vector potential) which enforces the constraint that the hole and spin currents cancel on average. One of the species (say holes) couples in addition to the external electromagnetic field. In a Hall-effect experiment the cancellation of hole and spin diamagnetic currents forces the fictitious field to acquire a finite average value:

$$B_{\text{gauge}} = -\frac{\chi_h}{\chi_s + \chi_h} B_{\text{ext}},$$

where $\chi_h$ and $\chi_s$ are the hole and spin diamagnetic susceptibilities and $B_{\text{ext}}$ is the external magnetic field. If the diamagnetic responses are temperature dependent, the gauge field is temperature dependent and therefore the Hall effect is temperature dependent. It is not yet clear whether the experimentally observed behavior can be recovered in detail but the theory does predict a significant temperature dependence of the diamagnetism.

A temperature-dependent Hall effect can also arise within a Fermi-liquid plus single-relaxation-time picture.\(^3\) Substantial dependence occurs if features of the Fermi surface which determine the Hall effect are washed out with increased temperature. There is no obvious connection in this model between Hall effect and diamagnetic susceptibility since $\chi_d$ is dominated by the response of states far from the Fermi surface; diamagnetism is not a Fermi-liquid property.\(^6\)

Lastly it has been shown\(^7\) that a weakly relaxation-rate-dependent Hall effect arises from failure of the single-relaxation-time approximation. Is there a corresponding result for $\chi_d$?

Two relevant issues will be tackled in this report. Firstly we formally derive the diamagnetic response of a charged-particle gas in the "dissipative regime" and show that the diamagnetism is proportional to the inelastic life-time $\tau$. Nagaosa and Lee\(^3\) advocate a picture in which bosonic hole degrees of freedom are scattered strongly by the fluctuating local spin chirality present in a uniform resonating-valence-bond state. In the normal state the resulting inelastic width $\tau^{-1}$ exceeds the coherence temperature $T_0$ associated with the holes. (We refer to this as the dissipative or incoherent regime.) No theory for the diamagnetic response of a system in the dissipative regime has been developed so far to our knowledge, yet it is the key ingredient in the gauge theory of the Hall effect.

Secondly, we will extend this calculation and consider the effect of anomalous current relaxation for short times; in other words the effect of corrections to the single-relaxation-time-scale approximation. The existence of a single-temperature-dependent relaxation timescale $\tau(T)$ is commonly assumed in transport theory, but this assumption is not supported by optical- and infrared-conductivity experiments in the case of the cuprate superconductors.\(^8\) In an earlier work we found the correction to the Hall effect in the presence of a high-frequency dissipative cutoff; here we find the analogous correction to $\chi_d$. It is found that the renormalization of $\chi_d$ is similar to that of the Hall number.

Since many-particle effects such as Bose versus Fermi statistics\(^9\) give only perturbative corrections in the dissipative regime it is sufficient to obtain the isolated-particle susceptibility $\chi_d^0$ and use $\chi_d = n \chi_d^0$. Inelastic scattering of an isolated particle is conveniently modeled by Caldeira-Leggett\(^10\) coupling to an oscillator bath. This model is a convenient and fundamental starting point because it can be fully diagonalized\(^11\) and the particle current decay is precisely exponential. In previous work\(^12\) we showed that the model leads to the usual drude expressions for the conductivities $\sigma^{\mu\nu}(\omega)$ ($\mu, \nu = x, y$) in the presence of a uniform magnetic field $H$ coupling to the particle. In fact this remains true independent of $\omega$, $\tau$ ($\omega$, $\tau$ is the cyclotron frequency, $eH/Mc$); the transport coefficients were found to be insensitive to details of the Landau-level structure.

The model consists of a particle bare mass $M$ to which is attached a large number of oscillators (masses $\mu_\alpha$, level
frequencies \( \omega_\alpha \) with infinite total mass. The bath “spectral density” defined as \( J(\omega) \equiv \sum_\alpha \mu_\alpha \omega_\alpha^2 \delta(\omega - \omega_\alpha) \) is chosen linear in frequency with coefficient to be identified as \( M/\tau \). In two dimensions we introduce independent sets of \( x \) and \( y \) oscillator baths, and a uniform magnetic field in the \( z \) direction: in the Landau gauge, \( \mathbf{A} = (0, Hx, 0) \). Since the center of mass of the complete system (particle plus bath) is fixed, one can replace the momentum and position operators in the particle kinetic-energy term by minus the total bath momentum and minus the bath center of mass. The Hamiltonian then reads (\( \hbar = 1 \))

\[
H = \frac{1}{2M} \left[ -\frac{e}{c} H Y_{\text{c.m.}} - P_{\text{tot}}^x \right]^2 + \frac{1}{2M} (-P_{\text{tot}}^y)^2 + H_{\text{osc}}^x + H_{\text{osc}}^y .
\]  

(1)

\( Y_{\text{c.m.}} \) is the \( y \) component of the bath center of mass, \( P_{\text{tot}}^{x,y} \) are the components of the total bath momentum. \( H_{\text{osc}}^x \) and \( H_{\text{osc}}^y \) represent independent sets \( x \) and \( y \) harmonic oscillators.

Having eliminated the particle degrees of freedom in this way the particle kinetic term induces a set of effective couplings between levels; the coupling of \( x \) and \( y \) oscillators disappearing in the limit of zero magnetic field. The Hamiltonian (1) is a quadratic form and can be diagonalized in terms of another set of independent harmonic oscillators with a different distribution of level frequencies. The somewhat messy details of this procedure can be found in Refs. 11 and 7. Specifically if one chooses initially a uniform distribution of oscillator levels \( \omega_\alpha = \Delta \omega \alpha, \alpha = 1, 2, \ldots, \infty \) [and corresponding masses \( \mu_\alpha \propto \omega_\alpha^2 \)] which together reproduce the required \( J(\omega) \) the new set of levels are the solutions of

\[
\cot \left( \frac{\omega}{\Delta \omega} \right) = 2(\omega \mp \omega_c) \tau ,
\]  

(2)

where the \( \mp \) sign refers to \( x \) and \( y \) labels. The solutions of these level equations are sketched in Fig. 1. Notice that when \( \tau \) is very large the solutions are \( \omega = \omega_\alpha \) as expected (except when \( \omega = \omega_c \)).

For the diamagnetic response we need to find the difference in the total energy between the case of finite and zero applied magnetic fields. The energy shift in an applied field \( \Delta E_H \) is the sum over zero-point-energy shifts of the oscillators which make up the bath. The total zero-point energies themselves are infinite but the difference is bounded above by \( \frac{1}{2} \hbar \omega_c \), the energy of a particle in the lowest Landau level. Notice from Fig. 1 that \( x \) and \( y \) levels shift in opposite directions in the applied field, the net energy shift being slightly positive in every case.

The final step is to let the level spacing tend to zero. \( \Delta E_H \) computed from Eq. (2) is plotted against the applied field in Fig. 2. In limiting cases (\( \hbar = 0 \))

\[
\Delta E_H \simeq \begin{cases} 
\frac{7 \omega_c^2 \tau}{4\pi}, & \text{when } \omega_c \tau \ll 1 , \\
\frac{\omega_c}{2} - \ln(\omega_c \tau) \quad 2\pi, & \text{when } \omega_c \tau \gg 1 .
\end{cases}
\]  

(3)

For large \( \omega_c, \tau \), the shift is linear in applied field and reduces correctly to the energy of the lowest Landau level

\[
\frac{1}{2} \omega_c \{ (a) \}
\]  

FIG. 2. The magnetic energy shift is given from (2) by

\[
\Delta E = \frac{1}{2\pi} \int_0^{\omega / \Delta \omega} \left[ \cot^{-1} \left( \frac{1 + 4\omega^2 \tau^2 - 2\omega \tau}{2\tau \omega_c} \right) - \cot^{-1} \left( \frac{1 + 4\omega^2 \tau^2 + 2\omega \tau}{2\tau \omega_c} \right) \right] d\omega .
\]

(a) \( \Delta E \) against \( \tau, \omega_c = 1 \). (b) \( \Delta E \) against \( \omega_c, \tau = 1 \).
as \( \omega_c \tau \to \infty \). Note that the diamagnetic susceptibility is not defined in this limit since one cannot take \( \omega_c \to 0 \) before \( \tau \to 0 \). For the relevant case of small \( \omega_c \tau \), the energy shift is quadratic in the applied field and proportional to the lifetime \( \tau \). Thus the first conclusion is that in the dissipative regime the temperature dependence of the diamagnetic susceptibility is the same as that of the inelastic lifetime \( \tau \).

Combining this with the transport results\(^7\) we have (\( \omega_c \tau \ll 1 \) and restoring \( \hbar \))

\[
\sigma_{xx} = \frac{n e^2 \tau}{M} \left( 1 + \omega_c^2 \tau^2 \right),
\]

\[
R_H = \frac{\sigma_{xy}}{H \sigma_{xx}} = \frac{1}{n e \tau},
\]

\[
\alpha \equiv \frac{\chi_d}{\sigma_{xx}} = \frac{7}{2\pi} \frac{\hbar}{M c^2} \cdot
\]

The results for the magnetoconductivity and Hall coefficient are of course well known and valid within classical transport theory. The diamagnetic susceptibility is not available classically (the diamagnetic response of a classical system vanishes identically) but the results (4) have been derived here on an equal footing. Note that the mass, lifetime, and carrier density in the dissipative regime are determined once \( \sigma_{xx} \), \( \sigma_{xy} \), and \( \chi_d \) are known.\(^{12}\)

Next consider the affect of corrections to a single-relaxation-timescale picture defined by \( J(\omega) = (M/\tau)\omega \) for all \( \omega \). The simplest way to study this is to introduce an upper cutoff \( \Omega_c \) on the bath levels which means that the particle motion is uncoupled from the bath at high frequencies and \( \sigma_{xx}(\omega > \Omega_c) \) vanishes.

In the presence of a high frequency cutoff \( \Omega_c \tau \gg 1 \) it can be shown that the level equation (2) is replaced by a level equation with renormalized lifetime \( \tilde{\tau} = \lambda \tau \) and cyclotron frequency \( \tilde{\omega}_c = \lambda^{-1} \omega_c \).\(^2\)

\[
\cot \left( \frac{\pi \omega}{\Delta \omega} \right) = 2(\omega + \Omega_c) \lambda \tau, \quad \omega < \Omega_c,
\]

\[
\lambda = 1 - \frac{1}{\pi \Omega_c \tau}.
\]

This is a good approximation far from the upper cutoff. For levels closer to \( \Omega_c \) there is an additional logarithmic correction factor appearing in the level equation but it can be checked that this makes a negligible contribution to the energy shift. Notice that \( \omega \tau \) is unrenormalized.

It is a fact that \( \sigma_{xx}(\omega = 0) \) is unaffected by the cutoff. However, there is a finite correction to \( \sigma_{xy} \) due to the downward renormalization of the lifetime; \( \sigma_{xy} \sim \omega_c \tau^2 \).

The cutoff affects the magnetic energy shift in two ways. Firstly, it removes the contribution of high-frequency levels to the total magnetic energy shift. But secondly, there is an enhanced energy shift coming from the upward renormalization of the cyclotron frequency. The former effect is negligible in comparison with the latter. The contribution of the total magnetic energy shift from the high-frequency levels is \( O((\Omega_c \tau)^{-2}) \) when \( \Omega_c \tau \gg 1 \). Inserting renormalized parameters in our earlier expression for the energy shift (3) gives an enhancement factor \( \lambda^{-1} \sim 1 + 1/\pi \Omega_c \tau \) which therefore dominates.

The factor \( \lambda \) is also the factor estimated earlier\(^7\) to be the enhancement of the hall number \( n_H \) in the presence of a cutoff. Thus we reach our second conclusion that in the dissipative regime a dissipative cutoff affects the ratio \( \chi_d/\alpha \) in a similar way to its effect on the Hall number:

\[
n_H \approx n_H^0 \left[ 1 + \frac{1}{\pi \Omega_c \tau} \right], \quad \chi_d = \frac{7}{4\pi} \mu_B^2 \left[ \frac{1}{\pi \Omega_c} \right].
\]

Some applications to cuprate superconductors. If holes are in the dissipative regime then it seems inevitable that the diamagnetism is strongly temperature dependent, unlike the usual Landau diamagnetism.\(^{13,14}\) In the gauge theories, Nagaosa and Lee\(^3\) and Ioffe and Kotliar\(^4\) note that the conductivity is dominated by the hole contribution and behaves as \( T^{-1} \). Therefore we find the hole susceptibility is proportional to inverse temperature. These authors in fact assume that the hole diamagnetic response is that of a Bose gas in the Boltzmann regime which is also proportional to the inverse temperature; so their conclusions are unchanged by our considerations.

A somewhat different connection between the Hall effect and diamagnetism is obtained in the cutoff model with \( \tau \sim T^{-1} \) from Eqs. (6). The temperature dependence of the Hall number in the cutoff model is definitely in qualitative accord with experiment and the model predicts that this should be associated with a constant contribution to the diamagnetic susceptibility. Obviously systematic studies of the diamagnetic contribution to the susceptibility in these systems are of great interest and serve to further constrain possible microscopic theories.

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\(^6\)A. Ishihara and M. Wadati, Phys. Rev. A 1, 318 (1970). These authors showed on the basis of a high temperature expansion that interactions reduce the diamagnetism of a Fermi gas below the Landau value.


Diamagnetism in the dissipative regime


The diamagnetic susceptibility can also be calculated at finite temperature where the zero point energies are replaced by harmonic oscillator free energies. In the limits \( \omega_c \tau \ll 1 \) this gives explicitly temperature dependent correction terms:

\[
\chi_d(T) = \chi_d(0) \left[ 1 + O(T^2 \tau^2) \right] \quad \text{for} \quad T \tau \ll 1 \quad \text{and} \quad \chi_d(T) = \chi_d(0) \times O(T \tau) \quad \text{for} \quad T \tau \gg 1.
\]

The transport coefficients are rigorously temperature independent.

The diamagnetic response of a two-dimensional Fermi gas is small at low temperature. R. Peierls, Z. Phys. 81, 186 (1933). This result does not apply of course in the dissipative regime where the Landau levels are inelastically broadened.