Shapiro steps on current-voltage curves of dc SQUIDs

C. Vanneste, a) C. C. Chi, W. J. Gallagher, A. W. Kleinsasser, S. I. Raider, and R. L. Sandstrom
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598

(Received 8 September 1987; accepted for publication 11 March 1988)

We have studied in detail the current-voltage (I-V) curves of dc SQUIDs in the presence of rf fields in the frequency range 2–18 GHz. Strong voltage locking at half-integral as well as the usual integral Shapiro step spacing is observed when the applied dc-magnetic flux is close to half-integral numbers of flux quanta. Numerical simulations are in excellent agreement with the experimental I-V curves. The half-integral steps are found to be associated with the flip-flop between two adjacent fluxoid states of the SQUID synchronized to the rf field.

I. INTRODUCTION

Because of their extreme sensitivity to magnetic flux, dc superconducting quantum interference devices (SQUIDs) have been most extensively investigated for low-frequency applications.1 Largely overlooked has been the fact that a dc SQUID, like a single Josephson junction, can be treated as a nonlinear current oscillator whose frequency is controlled by the voltage across its terminals. As such a dynamical system, the dc-current biased SQUID has recently been shown to display deterministic chaos in ranges of the junction parameters close to those actually used in low-frequency applications.2 A natural extension is to study the response of a dc SQUID to an external rf field. To our knowledge, such a problem has not been fully addressed in the past. We report a detailed study of the experimental current-voltage (I-V) curves of dc SQUIDs in the presence of rf fields.

II. EXPERIMENTAL RESULTS

For this study, we fabricated a set of dc SQUIDs consisting of a superconducting loop interrupted by two Nb-NbO2–PbInAu planar junctions individually shunted by a AuPd normal strip. Typical junction critical current, Ic, was ∼70 μA, shunt resistance R ∼2.5 Ω, and junction capacitance C<0.5 pF. The I-V curves were nonhysteretic because the corresponding Stewart–McCumber parameter,3,4 βc = 2eR2IcC/ħ was less than 1. The dimension of the SQUID loop ranged from 2.5×2.5 μm2 to 40×40 μm2, which was designed to cover a range of SQUID parameter βL = 2πLIc/Φ0 from 1 to 18, where L is the SQUID self-inductance and Φ0=ħ/2e is the superconducting flux quantum. The rf-frequency range used in this experiment was 2 to 18 GHz. Similar results having been observed throughout this frequency range, we will only present the results for 18 GHz, since they exhibited the clearest steps on the I-V curves.

Experimental I-V curves for a junction with βL ∼17 are displayed in Fig. 1(a). As expected, the experimental I-V curves present a periodic behavior as the magnetic flux in the loop increases. The first curve of Fig. 1(a) displays the I-V characteristics without microwaves for two values of the magnetic flux nΦ0 and (n + 1/2)Φ0 corresponding to maximum and minimum critical current, respectively. The following curves display how the I-V characteristic for a fixed value of the rf current5 evolves with increasing magnetic flux. For an integral number n of applied flux quanta (curve Φn = 0), only integral Shapiro steps are observed in the I-V curve. Such an I-V curve is very similar to the I-V curve of an rf-driven single Josephson junction.6 However, by progressively increasing the external magnetic field, a distinctively different feature starts to emerge: half-integral Shapiro steps between successive integral Shapiro steps (curve Φn = 0.17Φ0). The size of these half-integral Sha-

![Fig. 1. Experimental I-V curves at 4.2 K of two rf-driven dc SQUIDs: (a) one with a large inductance loop (βL ∼17) and (b) another with a small inductance loop (βL ∼1). The rf frequency is ν=18 GHz. The first curves in (a) and (b) display the I-V characteristics without rf power and at two successive values of the magnetic flux maximizing (full line) and minimizing (dashed line) the critical current. The following curves display the I-V characteristics at fixed rf power and three values of magnetic flux in the increasing order. For clarity, the successive curves are deliberately displaced horizontally.](http://jap.aip.org/)

---

a)Permanent address: Laboratoire de Physique de la Matière Condensée (U. A. 190), Université de Nice, 06034 Nice Cedex, France
pério steps increases and reaches a maximum when the magnetic flux increases to \((n + 1/2) \Phi_0\) (curve \(\Phi_\phi = 0.5 \Phi_0\)). The half-integral step can be as large as the size of the integral Shapiro steps. Further increasing the magnetic flux reduces the size of the half-integral steps. The original I-V curve recovers when the flux reaches the next integral flux quantum, and the whole pattern repeats upon further increase of the flux. It is worth pointing out that, while half-integral steps appear and disappear in between the integral steps as a function of the magnetic field, the overall shape of the I-V curve is only slightly changed and can be explained by the adiabatic model for single junctions.

A second set of experimental I-V curves for a junction with \(\beta_\phi \sim 1\) is displayed in Fig. 1(b). The first curve of Fig. 1(b) displays again the two I-V characteristics without microwaves with the applied flux, \(\Phi_\phi\), adjusted for the maximum and minimum values of the critical current. The larger modulation of the critical current reflects the smaller value of \(\beta_\phi\). The following curves display the I-V characteristics at a fixed microwave power for several values of the applied magnetic flux. There are two noticeable differences between this case and the previous one shown in Fig. 1(a): the half-integral steps only emerge for values of \(\Phi_\phi\) much closer to \(\Phi_\phi/2\), and the overall shapes of the I-V curves change substantially from one \(\Phi_\phi\) to another, in sharp contrast to the nearly identical overall shape in Fig. 1(a). Many other SQUIDs have also been examined, including some with different degrees of asymmetry in the critical current of the two junctions. When the critical-current asymmetry is large, the half-integral steps are always present on a certain portion of the I-V curve due to the bias-current induced \(\Phi_\phi\). Nevertheless, the pattern of the I-V curve is always periodic as a function of applied magnetic flux with period \(\Phi_0\).

III. NUMERICAL SIMULATIONS

Numerical simulations of the SQUID I-V curves have been performed by solving the following system of equations where each junction \(j\) \((j = a \text{ and } b)\) is described by the resistively shunted-junction (RSJ) model

\[
I_j = C_j \frac{dV_j}{dt} + \frac{V_j}{R_j} + I_d \sin \phi_j, \tag{1}
\]

\[
\frac{d\phi_j}{dt} = 2eV_j, \tag{2}
\]

\[
\phi_b - \phi_a = 2 \pi \Phi/\Phi_0, \tag{3}
\]

\[
\Phi = \frac{L}{2} \left( I_a - I_b \right) + \Phi_\phi, \tag{4}
\]

\[
I_a + I_b = I_0 + I_d \sin \Omega t, \tag{5}
\]

\[
V = V_j + \left( \frac{L}{2} \right) \frac{dI_j}{dt}, \tag{6}
\]

\(V_a, V_b, \text{ and } V\) are, respectively, the voltages across each junction and the total voltage across the SQUID. \(I_a\) and \(I_b\) are, respectively, the currents in the arms \(a\) and \(b\) of the SQUID, \(I_0\) is the dc current, and \(I_d\) is the rf-current amplitude. The circulating current in the loop is given by \(I_\phi = (I_a - I_b)/2\). For simplicity, the parameters of the two junctions are chosen to be identical \((I_{ca} = I_{cb}, R_a = R_b, \text{ and } C_a = C_b)\). The currents are normalized to the critical current of the SQUID \(I_j = I_j/(I_{ca} + I_{cb})\) and the frequencies to the characteristic frequency \(\omega_c = 2eR_jI_{c_j}/\hbar\).

In these units, the reduced time is \(\tau = \omega_c t\), the reduced voltage across each junction is \(v_j = V_j/R_j\), \(I_{c_j} = d \phi_j/dt\), the reduced total voltage is \(v = V/R_j\), \(I_{c_j} = v + (R_j/2) d1/dt\), and the circulating current in the loop is \(I_\phi = (I_a - I_b)/2\).

Figures 2(a) and 2(b) correspond to numerical I-V curves \((I_\phi, \text{vs-}(\nu))\) of SQUIDs with large and small loop inductance, respectively. The agreement between each of the numerical I-V curves and its experimental counterpart is excellent. In particular, the differences associated with the loop inductance are well reproduced by the numerical simulations. The only obvious deviations occur in the absence of microwaves, where some sharp resonant structures can be observed in the experimentlal I-V curves which are not due to the LC resonance of the SQUID loop. However, in the presence of microwaves, these structures are washed out and do not seem to modify the essential features of the half-integral steps.

IV. PHYSICAL INTERPRETATION

In order to understand the physical origin of the half-integral steps, numerical simulations of the time evolution of the voltage \(\nu\) and the circulating current \(I_\phi\) have been per-

![FIG. 2. Noise-free numerical I-V curves of two rf-driven dc SQUIDs: (a) one with a large inductance loop \((\beta_\phi = 12.75)\) and (b) another with a small inductance loop \((\beta_\phi = 0.85)\). The Stewart-McCumber parameter for each junction is \(\beta_\phi = 0.86\) and the reduced frequency is \(\Omega = 0.21\). The first curves in (a) and (b) display the I-V characteristics without rf power at \(\Phi_\phi = 0\) (full line) and \(\Phi_\phi = 0.5 \Phi_0\) (dashed line). The following curves display the I-V characteristics at \(I_\phi = 1.3\) and three values of magnetic flux in the increasing order. For clarity, the successive curves are deliberately displaced horizontally.

243 J. Appl. Phys., Vol. 64, No. 1, 1 July 1988

Vanneste et al., 243
formed. First, when the applied magnetic flux is exactly at an integral number of flux quanta $\Phi_0$, it is found that the phases $\phi_a$ and $\phi_b$ are synchronized ($\phi_a = \phi_b$), the loop current $i_L$ is exactly zero, and the observed evolution is identical to that expected for a single junction in the low-frequency regime.7

The link between the time evolution of the phase and the structure of the $I$-$V$ curve is detailed in Ref. 7 for a single junction and can simply be extended without modifications to the present case. Instead of repeating the arguments given in Ref. 7, we summarize the results. In this low-frequency regime ($\omega \ll \omega_c$), the SQUID can be considered as adiabatically following the time varying total current $i(t) = i_0 + i_1 \sin \Omega \tau$. The $I$-$V$ curves display three dc-current ranges similar to those of a single junction: in range I ($i_0 > i_1 + 1$), the $I$-$V$ curve asymptotically approaches the ohmic line; in range II ($i_1 - 1 < i_0 < i_1 + 1$), steps are located on a bump’ and are “stable,” i.e., they do not oscillate with the rf power; and in range III ($0 < i_0 < i_1 - 1$), steps are located along the ohmic line and oscillate with the rf power. It is not difficult to understand why the SQUID behaves like a single junction when $\Phi_2$ is equal to $n \Phi_0$. It is simply because the phases of the two junctions are locked together with zero phase difference between them. Therefore, Eqs. (1)–(6) can be reduced to the equations for a single junction.

To understand why the half-integral steps emerge on the SQUID $I$-$V$ curve for $\Phi_2$ close to $\Phi_0/2$, let us focus on range II where steps do not oscillate with the rf power. The order $n$ of a step in range II is given by the number $n$ of positive $2\pi$ rotations performed by the phase during each rf cycle for a single junction.7 For a SQUID with applied magnetic flux at integral number of flux quanta, the phases $\phi_a$ and $\phi_b$ are synchronized, and their time evolution is identical to that of a single junction, i.e., $\phi_a$ and $\phi_b$ are making the same number of $2\pi$ rotations per rf cycle. When the external flux deviates from an integral number of flux quanta, the fluxoid condition imposes a finite phase difference between $\phi_a$ and $\phi_b$ that makes the potential energy barrier between two adjacent fluxoid states smaller. With the help of the rf current, one would expect the SQUID to swing back and forth between two adjacent fluxoid states. To demonstrate that, Fig. 3 displays the time evolution of the voltages $v_a$ and $v_b$ across each junction for $\Phi_2 = \Phi_0/2$. The fixed value of the rf current and the values of the dc current have been chosen so that Figs. 3(a), 3(b), and 3(c) correspond, respectively, to the solutions for the step of order 3, the step of order 3.5, and the step of order 4 of the curve $\Phi_2 = 0.5 \Phi_0$ shown in Fig. 2(a).

In Figs. 3(a) (step of order 3), the voltages $v_a$ (full line) and $v_b$ (dotted line) exhibit three well-defined peaks in the fraction of the rf cycle where $i(t)$ is positive. These peaks are associated to the three $2\pi$ rotations that the phase $\phi_a$ and $\phi_b$ undergo during each rf cycle as expected on the third-order step. The solution is periodic with a period of one. Note the time shift between the two junctions and the nonzero value of the loop current $i_L$. If the dc-bias current is progressively increased, the solution becomes periodic with a period of two. As shown in Fig. 3(b), during the first rf cycle, the voltage $v_a$ exhibits four peaks while the voltage $v_b$ exhibits three peaks. This corresponds to four and three $2\pi$ rotations experienced, respectively, by $\phi_a$ and $\phi_b$. During the second rf cycle, the two junctions exchange their roles. The net result is an average voltage equal to $(3 + 4)/2 = 3.5$ per rf cycle in units of the Shapiro step spacing $\hbar \omega_c/2e$.

FIG. 3. Time dependence of the voltage $\mathrm{d}v_a/\mathrm{d}t$ and total bias current (full lines), $\mathrm{d}v_b/\mathrm{d}t$ (dashed line), and the loop current $i_L = (i_a - i_b)/2$ (mixed line) for increasing values of the dc current $i_0$. For clarity of the figure, the loop current was shifted by $-2$ and magnified by 10. The parameters used for this figure are $\beta_1 = 0.86$, $\beta_2 = 12.75$, the reduced frequency $\Omega = 0.21$, the rf current $i_r = 1.3$, and $\Phi_2 = 0.5 \Phi_0$: (a) $i_0 = 0.714$ (3rd-order step), (b) $i_0 = 0.857$ (3.5th-order step), and (c) $i_0 = 1.0$ (4th-order step).

The physical explanation can be given as follows. In the fraction of the first rf cycle where $i(t) > +1$, $\phi_a$ advances faster than $\phi_b$ because the loop current is in the same direction as the bias current for junction $a$ and in opposite direction for junction $b$. If $\phi_a$ advances fast enough, as is the case here, at the end of this fraction of the rf cycle, $\phi_a$ will have one more $2\pi$ revolution than $\phi_b$, making it energetically favorable for the SQUID system to switch to the adjacent fluxoid states when the SQUID returns to the zero-voltage state during the remaining fraction of the rf cycle. Thus, when the next rf cycle begins, the roles of $a$ and $b$ are just reversed. The switching of the fluxoid states is more clearly demonstrated by the time evolution of the loop current $i_L$ is shown by the dash-dotted line in Fig. 3(b). If the dc-bias current is increased further [Fig. 3(c)], a new period one solution takes place such that both phases $\phi_a$ and $\phi_b$ undergo four $2\pi$ rotations per rf cycle. This solution is similar to the solution of Fig. 3(a) except that it corresponds now to the fourth-order step. By increasing the dc current again, the same evolution of the solutions repeats, giving rise to successive integral and half-integral Shapiro steps on the $I$-$V$ curve. This flip-flop mode of a dc SQUID has been previously reported in the contexts of self-induced LC resonance structures on the SQUID $I$-$V$ curves8 and flux shuttling in SQUID digital electronics.10

The overall shapes of $I$-$V$ curves shown in Figs. 1(a) or 2(a) for different amounts of external flux do not differ from each other very much. In fact, they are very similar to the $I$-$V$ curve of an rf-driven single junction with the same junction
parameter. This can be explained by the same adiabatic model developed for the single junction and the fact that the critical current of the SQUID does not change much for different values of $\Phi$. In contrast, the overall shapes of $I$-$V$ curves for the SQUID with a small $\beta_L$ shown in Figs. 1(b) or 2(b), do change substantially with the external flux. When the critical current is near its maximum value with small external flux, the overall shape can still be explained by the adiabatic model. However, when the critical current becomes much smaller, and the external flux is near $\Phi_o/2$, the overall shape of the $I$-$V$ curve differs substantially from the adiabatic model. We note that the corresponding characteristic frequency $\omega_c = 2eRI_c/\hbar$ is reduced with the critical current and becomes smaller than the rf frequency (18 GHz). Therefore, the adiabatic model is no longer applicable. In fact, the overall shape of the $I$-$V$ curve is indeed similar to that of a single junction in the high-frequency regime ($\omega > \omega_c$). 11

V. CONCLUSION

To conclude, experimental $I$-$V$ curves of rf-driven dc SQUIDs display strong voltage locking at integral values of the Shapiro step spacing $2\omega / 2e$, like the $I$-$V$ curves of rf-driven single junctions when the external flux is near an integral number of flux quanta, but can also exhibit well-defined half-integral Shapiro steps when the external flux is close to $\Phi_o / 2$. Through numerical simulations, the half-integral Shapiro steps have been found to be associated with the flip-flop between two fluxoid states of the SQUID synchronized to the rf field. This behavior has been observed for a large range of loop inductances and rf frequencies for overdamped SQUIDs ($\beta_c < 1$). It would be interesting to undertake a similar study of the underdamped SQUIDs ($\beta_c > 1$) where more complicated regimes are likely to occur. Finally, we note that the occasional appearance of half-harmonic steps in the $I$-$V$ curves of rf-driven Josephson junctions arrays has been observed, 12 which might be the result of this flip-flop mode because of the effective SQUID loops in the sample. It would be interesting to check the magnetic field dependence of the amplitudes of the half-harmonic steps in such arrays to see whether it is consistent with the flip-flop mechanism.

ACKNOWLEDGMENTS

One of us (C. V.) would like to thank Dr. A. P. Malozemoff and Dr. C. C. Tsuei for their hospitality during his visit at the IBM Thomas J. Watson Research Center.

5The absolute external rf power was not measured. Instead, the rf current in the junction was estimated by comparing the $I$-$V$ curves with the predictions of the adiabatic model introduced in Ref. 7.
6This was experimentally confirmed by looking at the $I$-$V$ curves of single junctions especially designed with the same parameters as the junctions of the SQUID and located on the same chip. We have checked that, unlike the SQUID's $I$-$V$ curves, the $I$-$V$ curves of the single junctions were not sensitive to the external magnetic field.
8We do not know at the moment the physical origin of the sharp structures observed in the $I$-$V$ curves of our SQUIDs without microwaves. The same structures were also observed in the $I$-$V$ curves of single junctions. However, for SQUIDs, these structures display a periodic behavior as a function of the magnetic flux, thus indicating a phase-dependent mechanism.
11This result was checked by the comparison with numerical $I$-$V$ curves of single junctions in the high-frequency regime.
12J. Niemeyer, private communication.