

國立清華大學 100 學年度碩士班入學考試試題

系所班組別：數學系碩士班應用數學組

考試科目（代碼）：高等微積分(0201)

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Advanced Calculus Written Exam

1. (18%) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n^2$ converges. Are the following series convergent? Prove your answer or provide a counterexample.

(a) $\sum_{n=1}^{\infty} \sin(a_n)$ (b) $\sum_{n=1}^{\infty} \frac{a_n}{n}$

2. (18%) Let A be an open subset of \mathbb{R}^n . Prove or disprove, by giving a counterexample, the following statements.

(a) $\text{int}(\overline{A}) = A$.

(b) $A \cap \overline{B} \subset \overline{A \cap B}$ for any $B \subset \mathbb{R}^n$.

(Here “int” means interior, upper bar means closure.)

3. (12%) Suppose $a_n \in \mathbb{R}$ and $|a_n| \leq \frac{n^2}{2^n}$ for every positive integer n . Let

$$f(x) = \sum_{n=1}^{\infty} a_n x^n, \quad f_k(x) = f\left(x + \frac{1}{k}\right).$$

Show that f_k converges uniformly to f on $[-1, 1]$ as $k \rightarrow \infty$.

4. (12%) Let $f(x, y) = \sin\left(\frac{1}{|x-y|^2}\right)$ when $x \neq y$ and let $f(x, x) = 0$ for any x . Is this function Riemann integrable on $[0, 1] \times [0, 1]$? Prove your answer.

5. (12%) Define $f: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n$ by $f(x) = \frac{x}{\|x\|}$. Calculate the derivative $Df(x)$ for f at $x \in \mathbb{R}^n \setminus \{0\}$.

6. (14%) Show that the solution set for the system

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ x^3 + y^3 + z^3 = 0 \end{cases}$$

is a smooth curve (i.e. a curve of class C^∞) in \mathbb{R}^3 .

7. (14%) Evaluate the double integral

$$\iint_E e^{x^2+2xy+5y^2} dA$$

where E is the ellipse $\{(x, y) \in \mathbb{R}^2 : x^2 + 2xy + 5y^2 \leq 1\}$.