AVERAGE DISTANCE STRUCTURES OF TRELLIS CODES

Chi-chao Chao
Department of Electrical Engineering
National Tsing Hua University
Hsinchu, Taiwan 30043, R.O.C.

Abstract

In this paper we use random coding analysis to study the average distance properties of trellis codes. The generating function for the average number of error events with each distance from the transmitted code path in the ensemble of random trellis codes is fully determined.

Summary

The average distance structure of binary convolutional codes is determined in [1]. In this paper we generalize the random coding analysis to trellis codes. The model for trellis codes considered consists of a convolutional encoder and a signal point selector, where n uncoded binary bits are first convolutionally encoded to n + r coded bits, and then the n + r bits are used to select one signal point out of the 2^r + point signal constellation. Other models can be considered as special cases of the model here.

Let T_k be the number of error events with distance d from the transmitted code path (averaged over all possible code paths.) For the additive white Gaussian noise (AWGN) channel, the distance metric used is the squared Euclidean distance. The transfer function T(D) is the generating function of T_k.

\[ T(D) = \sum d^k T_k D^d. \]

In principle T(D) can be computed from a product state diagram with 2^r states [2] [4], where 2^r is the number of corresponding convolutional encoder states. For trellis codes with the so-called UZV symmetry (Ungerboeck [5], Zehavi and Wolf [6], T(D)) can be found from a state diagram with 2^2 states [6]. The transfer function is of great importance in performance evaluation since the first event error probability P_E for maximum likelihood decoding can be upperbounded by the well-known transfer function bound. For example, for trellis codes on AWGN channels, we have [6] [7]

\[ P_E \leq Q \left( \frac{\Delta_{\min} E_b}{N_0} \right)^{1/2} \exp\left( -\frac{E_b}{N_0} S(D) \right), \]

where \( E_b/N_0 \) is the symbol-to-noise ratio, \( \Delta_{\min} \) is the minimum (squared Euclidean) distance between different code paths, and \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-t^2/2) dt \).

We now consider the ensemble of trellis codes whose convolutional encoders are time-varying. Consider only feedforward encoders. A time-varying convolutional code [8] has generator polynomials which may be changed after each time unit. A uniform probability measure is imposed by randomly resitting the tap positions of modulo-2 adders in the shift register encoder after each shift. Here a random binary vector is also added to the outputs of modulo-2 adders, this is required to let the output code path of all-zero input be different from other code paths. Now the output of the overall trellis encoder will be a random signal point with uniform probability. The ensemble of time-varying trellis codes considered here includes the ensemble of ordinary fixed trellis codes as a subset. Suppose the \( (n+r,n) \) convolutional encoder has memory \( v_1, v_2, \ldots, v_n \) (and total memory \( \nu = v_1 + \cdots + v_n ) \). Let \( T_{d_k} \) be the average number of error events with distance \( d \) from the transmitted code path in the ensemble of such time-varying trellis codes. Given a \( 2^{n+r} \)-point signal set \( S \), let \( S(D) = 2^{-n-r} \sum_{s \in S} D^{(s)} \), where \( s, s' \) are any signal points in \( S \) and \( d(s, s') \) is their distance. Our result shows that the generating function of \( T_{d_k} \) satisfies

\[ T_{d_k} = T(D)^k, \]

where \( T(D) = F(L) |_{L=S(D)} \), which is the \( k \)-th power of the generating function of \( T(D) \).

The function \( F(L) \) can be found as follows. Suppose we put \( v_1, v_2, \ldots, v_n \) in decreasing order and group equal \( v_i \)'s into, say, \( x \) subsets. Let \( e_1, f_1 \) be the differences between consecutive subsets. For example, if \( v_1 = 4, v_2 = 3, v_3 = 1, v_4 = 1, \) then \( (e_1, f_1) = (1, 1), (e_2, f_2) = (1, 2) \). Then \( F(L) \) may be computed from the following algorithm:

1. \( F(L) = 1; \)
2. \( F(L) = F(L) + (2^r-1)L; \)
3. \( F(L) = L^k F(L) + (1 - F(L) \sum_{l=0}^{r-1} L^l); \)

Suppose the signal set \( S \) contains all binary \( (n+r) \)-tuples and the distance metric is Hamming distance. Then the resulting trellis code is just an ordinary convolutional code. We have \( S(D) = 2^{-n-r} (1 + D)^{2^r} \) and \( T(D) = F(D) (1 + D)^{2^r} \), which gives a generalization of the result in [2], where only the case \( v_1 = v_2 = \cdots = v_n \) is considered. Our result can have several applications. First, the inequality

\[ \sum_{0 \leq d \leq d_{\max}} T_d \geq 1 \]

gives a Gilbert-type lower bound on minimum distance. This bound can be computed from the known \( T(D) \) and it holds for any trellis code satisfying the model considered. Work is still in progress in investigating its asymptotic behavior and comparing the bound with that in [9].

Consider signal constellations in which all possible distances are integral multiples of the minimum distance between signal points, e.g., rectangular signal constellations. Suppose the minimum distance between signal points is normalized to 1. Define the dominant root \( \alpha \) by

\[ \alpha = \lim_{d \to \infty} \frac{S(D)}{D^d} \]

\( \Delta_{\min} \) is defined as the reciprocal of the least-magnitude root of the denominator of \( T(D) \) and it tells us the growth rate of the number of error events at large distance. The corresponding dominant root \( \alpha \) for \( T(D) \) can give us a prediction of the growth rate of error events for trellis codes.

Finally, by the transfer function bound (1), we can use \( T(D) \) to study the ensemble performance of trellis codes with different numbers of states and/or different signal constellations.

References


