關於線紋面之一定理

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余作「四曲面組」[1]中,設 Sy_i $(i=1,2,3,4)$為四曲面,其方程式順次為

$$(1)\qquad y_i^{(1)}=y_i^{(1)}(u,v),\ y_i^{(2)}=y_i^{(2)}(u,v),\ y_i^{(3)}=y_i^{(3)}(u,v),\ y_i^{(4)}=y_i^{(4)}(u,v)\qquad (i=1,2,3,4).$$

則 u, v 均同值之四點,謂之相當點。若任四相當點 Py_i $(i=1,2,3,4)$ 不在一平面上,則

$$(2)\qquad \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & y_1^{(3)} & y_1^{(4)} \\ y_2^{(1)} & y_2^{(2)} & y_2^{(3)} & y_2^{(4)} \\ y_3^{(1)} & y_3^{(2)} & y_3^{(3)} & y_3^{(4)} \\ y_4^{(1)} & y_4^{(2)} & y_4^{(3)} & y_4^{(4)} \end{vmatrix} = 0.$$

此四曲面組之影性 (Projective properties), 可藉適合於完全積分條件 (Integrability conditions)

$$(3)\qquad \frac{\partial a_{ij}}{\partial v}+\sum_{k=1}^{4} a_{ik}b_{kj}=\frac{\partial b_{ij}}{\partial u}+\sum_{k=1}^{4} b_{ik}a_{kj}\qquad (i,j=1,2,3,4)$$

之微分方程式

$$(4)\qquad \frac{\partial y_i}{\partial u}=\sum_{j=1}^{4} a_{ij}y_j,\qquad \frac{\partial y_i}{\partial v}=\sum_{j=1}^{4} b_{ij}y_j\qquad (i=1,2,3,4)$$

以研究之。

一「二次曲面」與此四曲面相切於一組相當點 Py_i $(i=1,2,3,4)$ 者,謂之四曲面組於相當點 Py_i 之「切二次曲面」,若在相

當點Py_i,此四曲面組有一切二次曲面,則下之方程式,亦能在相當點Py_i適合:

$$(5)\quad \begin{aligned} &d_{23}d_{34}d_{42}-d_{32}d_{43}d_{24}=o,\\ &d_{34}d_{41}d_{13}-d_{43}d_{14}d_{31}=o,\\ &d_{41}d_{12}d_{24}-d_{14}d_{21}d_{42}=o,\\ &d_{12}d_{23}d_{31}-d_{21}d_{32}d_{13}=o, \end{aligned}$$

而

$$(6)\quad \begin{aligned} &d_{12}=a_{13}b_{14}-a_{14}b_{13}, && d_{13}=a_{14}b_{12}-a_{12}b_{14}, && d_{14}=a_{12}b_{13}-a_{13}b_{12},\\ &d_{21}=a_{23}b_{24}-a_{24}b_{23}, && d_{23}=a_{24}b_{21}-a_{21}b_{24}, && d_{24}=a_{21}b_{23}-a_{23}b_{21},\\ &d_{31}=a_{32}b_{34}-a_{34}b_{32}, && d_{32}=a_{34}b_{31}-a_{31}b_{34}, && d_{34}=a_{31}b_{32}-a_{32}b_{31},\\ &d_{41}=a_{42}b_{43}-a_{43}b_{42}, && d_{42}=a_{43}b_{41}-a_{41}b_{43}, && d_{43}=a_{41}b_{42}-a_{42}b_{41}, \end{aligned}$$

逆之亦然。　(5)之四方程式中,任其一可由餘三方程式得之。

按威而秦司克(Wilczynski)之理,[2]若以適合於完全積分條件

$$(7)\quad \begin{aligned} &a'_{uu}+g_u+2ba'_v+4a'b_v=o,\\ &b_{vv}+f_v+2a'b_u+4ba'_u=o,\\ &g_{uu}-f_{vv}-4fa'_u-2a'f_u+4gb_v+2bg_v=o \end{aligned}$$

之微分方程式

$$(8)\quad \begin{aligned} &y_{uu}+2by_v+fy=o,\\ &y_{vv}+2a'y_u+gy=o \end{aligned}$$

研究一曲面S_y之影性,則y, y_u, y_v, y_{uv}為(8)之「半不變式」(Semi covariants),而Py, Py_u, Py_v, Py_{uv}之軌跡,乃為S_y之「半不變四曲面組」。若令

$$(9)\quad y_1=y,\quad y_2=y_u,\ y_3=y_v,\ y_4=y_{uv},$$

則此半不變四曲面組之微分方程式為

(10)
$$\frac{\partial y_1}{\partial u}=y_2,$$
$$\frac{\partial y_1}{\partial v}=y_3,$$
$$\frac{\partial y_2}{\partial u}=-fy_1-2by_3,$$
$$\frac{\partial y_2}{\partial v}=y_4,$$
$$\frac{\partial y_3}{\partial u}=y4,$$
$$\frac{\partial y_3}{\partial v}=-gy_1-2a'y_2,$$
$$\frac{\partial y_4}{\partial u}=(2bg-f_v)y_1+4a'by_2-(2b_v+f)y_3,$$
$$\frac{\partial y_4}{\partial v}=(2a'f-g_u)y_1-(2a'_u+g)y_2+4a'by_3.$$

由是得

(11)
$$d_{12}=o,\quad d_{13}=o,\quad d_{14}=1$$
$$d_{21}=-2b,\quad d_{23}=f,\quad d_{24}=o,$$
$$d_{31}=2a,\quad d_{32}=-g,\quad d_{34}=o,$$
$$d_{41}=16a'^2b^2-(2a'_u+g)(2b_v+f),$$
$$d_{42}=-(2b_v+f)(2a'f-g_u)-4a'b(2bg-f_v),$$
$$d_{43}=-(2a'_u+g)(2bg-f_v)-4a'b(2a'f-g_u),$$

以(11)諸值代入(5),則得

(12)
$$a'd_{43}=o,$$
$$bd_{42}=o,$$

此為半不變四曲面組在相當點有一切二次曲面之必要與

必然條件

若 $a'=o$，則 $g_u=o$，

$\therefore d_{42}=o$

同理 $b=o$，則 $d_{43}=o$. 但 $a'=o$ 或 $b=o$ 爲 Sy 係一線紋面 (Ruled surface) 之必要與必然條件。故得

定理. 線紋面之半不變四曲面組，在任何相當點，有一切二次曲面。

此定理之逆理不常眞

註(1) D. Sun, Projective Differential Geometry of Quadruples of Surfaces with Points in Correspondence, Tohoku Mathematical Journal, 印刷中

註(2) E. J. Wilczynski, Projective Differentical Geometry of Curved Surfaces, First Memoir, Transactions of American Mathematical Society, vol. 8, 1907, pp, 233-260.

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