ON FINITE SYSTEMS

by

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1

The science of logic may be conveniently studied under three divisions, which treat of :

(1) The realm of the finite,

(2) The realm of the transfinite, and

(3) The realm of the indefinite, or the paradoxical realm.

The first among these, the realm of the finite, forms so close a field of its own that most if not all of the problems touching the field seem capable of solution independently of the difficulties which harass the other two realms. It may be a point of wisdom for the logician to direct his attention for a time to this field alone.

The subject matter of the present paper is a sequel to Dr. Sheffer's *Notational Relativity*. It is a research in the finite realm. Accordingly, some of even the most general statements made here will require correction if their application is to be extended to the realm of the transfinite.

For purposes of mathematical analysis it is useful to adopt a modified form of Russell's Theory of Types. It can be shown that any finite system can be thrown into a form accordant with the modified theory, though such a form is not strictly necessary. We shall first explain this theory before commencing on our discussion.

A type is the range of values which a variable involved in a function can significantly assume. We shall introduce the arbitrary but convenient principle that two types which do not completely coincide are always mutually exclusive. Any object belongs to one and only one type. In the finite realm we may, theoretically, use functions only of one variable, thus avoiding the introduction of arbitrary serial order. We shall think of a function $R\hat{x}\hat{y}$ which involves two variables as a function involving only one variable but having values which are again functions of one variable. Such values will be called immediate values. Thus we think of $R\hat{x}\hat{y}$ as having immediate values of the form $Ra\hat{y}$, or, $P\hat{y}$. The expression "Rab" means $(Ra\hat{y})b$; i.e., (Ra)b. When we reach values which are no longer functions, we call them final values. For example, if "Rab" is a proposition, then Ra is an immediate value for R, Rab an immediate value for Ra and also a final value for Ra as well as for R. In general a function R or $R\hat{x}\hat{y}\hat{z}$... will be considered as having immediate values of the form $Ra\hat{y}\hat{z}$..., which is written "Ra."

Values (immediate) for one and the same function must all belong to the same type. Two functions are of the same type when and only when their (immediate) values are of the same type—call it T—and their (immediate) variables have the same type—call it T', which may either be identical with or different from T—as their range of significance. Thus the type of a function is defined by two types, a value type T and an argument type T'. The type of the function is then denoted by TT'. For instance, if T_0 is the type of $\sim p$, then, since p also belongs to T_0, $\sim\hat{p}$ or $\sim$ belongs to T_0T_0.

Types of functions are therefore said to be derivative. Types which are not derivative are primitive. In any system there is always the type of truth-values, the primitive logical type. A symbol, simple or complex, whose object belongs to this type is called a proposition. Only such a symbol standing alone has assertive significance. All other primitive types in the system will be non-logical. They might be called types of individuals. A function whose final values belong to the primitive logical type is called a propositional function, whereas one whose final values belong to a primitive non-logical type is called a descriptive function. When the number of primitive types in a system is finite, the number of all types in the system is Aleph-0.

The following three principles will be assumed, which I call the extensional postulates :—

(1) Any two objects a and b such that for any R, if "Ra" is a true proposition "Rb" is also true, are identical. It is important to note that here the R need not be mentionable in the system, though of course its type must be in the system.

(2) If a proposition "p" occurs in another proposition "q", the substitution of a third proposition "r" for "p", if "r" and "p" are either both true or both false, does not affect the truth or falsehood of "q".

(3) Any two functions R_1 and R_2 such that $R_1 a$ is always identical with $R_2 a$ for any a which makes "$R_1 a$" significant, are themselves identical. If, therefore, the number of objects in a type T is μ, and that in T'is ν, the number of functions in TT' is μ^{ν}.

From the above principles it follows that there are two and only two objects in the type of truth-values. These two truth-values are called truth and falsehood. We can say, for example, either that "Ra" is a true proposition, or that Ra is truth. But we may also use the ambiguous phrase "is true" to cover both expressions.

2

Some preliminary remarks concerning the nature of variables will not be out of place. The vicious ambiguity inherent in a sign like "$\phi\hat{x}$" is seen if we consider the expression "$\phi(\phi\hat{x})$". Here the scope of the cap "^" may either stop before the first "ϕ" or between the two "ϕ"s. In the latter case the expression, which is better written "$\phi\phi$", violates the principle of types. In the former case, however, there is no violation of types. The expression can be written "$\hat{x}\{\phi(\phi x)\}$". Thus we can certainly have $\sim(\sim p)$. The curious notation which we have now introduced always definitely indicates what we may call the "scope" of a variable (not to be confused with the "range of significance" (type) of a variable, which is entirely another matter). For example, when we see in "$\hat{x}\{\phi(\phi x)\}$" that the expression "$\phi(\phi x)$" is all that is included within the brackets immediately following "$\hat{x}$", we know that the scope of the variable "x" is the expression "$\phi(\phi x)$". The scopes of two variables either lie completely outside of each other or the one includes the other without being included by it. (Since we do not form functions of more than one (immediate) variable, the scopes of two variables never completely coincide.) When the scope of one variable includes the scope of another variable, it is necessary to use different letters for the two variables, though these may have the same type as their range of significance.

An expression like "$\hat{x}\hat{y}(Ryx)$" which defines the converse of R, is to be read "$\hat{x}\{\hat{y}(Ryx)\}$", or, more elaborately, "$\hat{x}[\hat{y}\{(Ry)x\}]$". (It is evident that $\hat{x}\hat{y}$ (Rxy) is the same as R, $\hat{x}$(Rx)a the same as Ra, $\hat{x}\hat{y}$(Rxy)ab the same as Rab, $\hat{x}\hat{y}$(Ryx)ab the same as Rba, etc., whenever the expressions are significant.)

It frequently happens (in fact, this has been happening all the time in this paper) that in a discussion not entirely in symbolic language the scope of a variable may extend beyond the symbolic part into the English of the discussion. The variable is then called an undetermined constant, or a parameter, or a real variable, the terms all having the same meaning. "Real variable" is not an inappropriate designation; for while the effect of ambiguity of a variable whose scope does not extend beyond the symbolism must somewhere be annulled in the symbolism, the effect of ambiguity of an undetermined constant pervades the whole symbolism. It is clear that in a complete symbolic genuine assertive context there can be no real variable. But in a "postulational" system which admits of various empirical (or other) interpretations the concepts which are thus variously interpreted are always real variables, which as such cannot be made apparent within the system, since their scopes do not stop anywhere in the system. These variable concepts must be treated in the system as if they were constants. The system in itself is not genuinely assertive.

Suppose we say, "Any value $\sim$q of the function $\breve{p}(\sim p)$," etc. It is obvious that the symbol "$\sim$q" does not denote any particular value, and is consequently as bad as "$\breve{p}(\sim p)$." However, we have made no mistake. Since "q" is a variable, the symbol "$\sim$q" is of course functional; but here the variable "q" is made apparent only in the English, and should therefore be treated as a constant within the symbolism.

We sometimes say, "If we have a constant R, then this R," etc. Here "R" is clearly a variable. But it is entirely correct to say "constant"; for "R" is evidently the variable of a constant (more accurately, a variable whose values are constants). "*If* we have a constant R," that is to say, "for *any* constant R," etc. In an actual case R would be constant, and we are saying, "*If* we have an actual case," etc. It is therefore wrong to say here, "If we have a variable R"; for then "R" becomes the variable of a variable, which is not what we meant in the original clause.

These remarks serve to remove a possible confusion or misunderstanding. They explain why I deviate from the style of Dr. Sheffer.

3

A primary system is an assertive theory about the universe or some portion (aspect) of reality. The logical form of a primary system is called a primary system-form. I use the term "logical form" in a very technical sense. It refers to the logical product of all that can be said in logical terms about something. We can say, for example, that a cardinal number is the logical form of a class. This way of defining a cardinal number has an advantage. If Aleph-O is considered as a class of classes, it may happen that it is a null class. In this case Aleph-l must also be a null class, and therefore identical with Aleph-O. But as logical forms, Aleph-l is distinct from Aleph-O, in the sense that logic cannot prove that the class of classes defined by Aleph-O is null and (therefore) identical with that defined by Aleph-l. Logical forms, thus understood, are transcendental concepts.

Euclidean space (or Euclidean geometry) may be taken either as a genuine (true or false) primary system, or as a primary system-form. In the latter case, it is not genuinely assertive

A typified system is one way of conceiving a primary system, a way in which the primitive types are specified. The logical form of a typified system is called a typified system-form. We shall later introduce the Shefferian term "tropicities" to cover an important species of typified system-forms; namely, those which are "exhaustive" and "monofundamental." Euclidean geometry in which points form the only primitive non-logical type may be taken as an example either of a typified system or of a (transfinite) tropicity, a typified system-form. Another tropicity, different though "equivalent," is Euclidean geometry in which spheres form the only primitive non-logical type. The latter (not the former) will be referred to as the Huntingtonian tropicity.

A basalized system is one way of conceiving a typified system, a way in which all the concepts in the system are defined or made definable in terms of specified primitive concepts (with the aid of logical concepts). If two (constant) expressions in the system, "a" and "b", simple or complex, representing objects belonging to the same type, are such that "a=b" is recognized as true by (asserted in or deducible from) the system, they are said to mean the same concept. Otherwise, they are said to mean distinct concepts. When the latter is the case, it does not follow that "a=b" is false, unless the system is "exhaustive." In the case of an "exhaustive" system "concept" is synonymous with "object mentionable in the system." In defining a concept, the definition may either be direct or indirect. An indirect definition is of the form "$a=(\imath x)(\phi x)$ Df". It is allowed when and only when "E! $(\imath x)(\phi x)$" is recognized as true by the system. From the extensional point of view, the distinction between the two kinds of definition is not as absolute as Russell seems to have thought it; for it is relative to the particular way of basalization.

The logical form of a basalized system is called a basalized system-form, or a base-form. A base-form is not to be conceived as a language which allows contradictory systems to be built upon it. For it also determines the assertions; and that completely, if the system-form is exhaustive. In the Huntingtonian base-form (or basalized system, depending on which way one takes it) there is only one primitive (non-logical) concept, "includes", which is of the type $(T_0 T_1) T_1$, or, $T_0 T_1 T_1$, where T_0 is the logical type of truth-values, T_1 the non-logical type of spheres. The Huntingtonian base-form is a special way of conceiving the Huntingtonian tropicity; for it is evident that we can choose other primitive concepts while preserving the primitive type of spheres.

How the primitive logical concepts are specified will not be considered as affecting the basalized system. In particular, we may specify the following three as the primitive logical concepts:—

(1) $\sim$, of type $T_0 T_0$.

(2) $\square$, of type $T_0 T_0 T_0$. "$\square pq$" means p.q. We may use the abbreviation "$\square pqr$" for "$\square(\square pq)r$", which means p.q.r.

(3) $\sqcap$, of the ambiguous type $T_0(T_0 T)$, or, $T_0 . T_0 T$, where T may be any type, primitive or derivative, logical or non-logical. "$\sqcap\phi$" means $(x). \phi x$. Thus, "$\sqcap\dot{x}(\phi xx)$" means $(x). \phi xx$; etc. We may use the abbreviation "$\sqcap\dot{x}\dot{y}(\phi xy)$" or "$\sqcap\phi$" for "$\sqcap\dot{x}$

{⊓(ϕx)}", which is the same as "⊓ẍ {⊓ÿ (ϕxy)}", meaning (x, y). ϕxy. (A letter with dieresis always goes with the following expression.)

As an example of (direct) definition we may give the following definition of identity:—

$$I = \ddot{a}\ddot{b}[\sqcap \phi\{\sim[\square(\phi a)\{\sim(\phi b)\}]\}] \text{ Df},$$

for which we better adopt the following mode of writing:—

$$I \text{ Df } \ddot{a}\ddot{b}\{\sqcap \phi(\sim:.\square:\phi a:\sim.\phi b)\}.$$

"Iab" means a = b.

It is convenient sometimes, in order to avoid ambiguity, to write the name of the type which is the range of significance for a variable as a subscript to the variable when it first appears (in a complex expression); thus, "$\ddot{p}T_0$". We may also introduce the notation "|T|" as an abbreviation for "($\imath\phi T_0 T$) ($\sqcap\phi$)". In virtue of this definition, which is allowable only if we assume the extensional postulates, we see that "|T|a" means "a belongs to the type T." We can say, for example, "$|T_0 T_0|\sim$". If T_1 is the type of spheres, $|T_1|$ is the class of all spheres. A proposition of the form "|T|a" is always true whenever significant.

There are certain concepts which can be defined without using any primitive concept. For example, the identical operation can be defined as ẍ (x). "Is true" can be defined as $\ddot{p}\, T_0$ (p), which is the identical operation for the type T_0. Another example is :—

$$\text{CvDf}\ddot{R}\ddot{x}\ddot{y}\ (Ryx).$$

"CvR", as we see, denotes the converse of R. This last example also shows a certain advantage of our one variable view of functions, in which the propositional function Cv can serve beautifully the purpose of a descriptive function.

A codified system, or a code, is one way of conceiving a basalized system, a way in which all the propositions recognized as true by the system are demonstrable from specified primitive propositions with the aid of logic. The theory of inference presents some weird problems which we shall not discuss here. The logical form of a codified system (code) is called a codified system-form, or a code-form. The Huntingtonian postulates for geometry form a code or code-form, which is a special way of conceiving the Huntingtonian base-form. For it is clear that a different set of postulates may be chosen, built on the same primitive type (variable) and the same primitive concept (variable) as in the Huntingtonian code-form, spheres and inclusion, such that the one set of postulates follows from the other, and conversely.

One may also conceive a basalized system as a set of codes, a typified system as a set of basalized systems, and a primary system as a set of typified systems. Systems are equivalent if they finally come under the same primary system. Analogous remarks apply to system-forms.

Again, a primary system-form may be considered as a set of primary systems which have the same logical form but are not equivalent among themselves, since they may be different theories concerning different aspects of the universe. The relations between codes, code-forms, and primary systems which all come under the same primary system-form, are represented in the following graph :—

Primary system-form α

	Primary system I	$\not\equiv$	Primary system II	$\not\equiv \cdots$
Code-form 1	Code A_1	$\not\equiv$	Code A_2	$\not\equiv \cdots$
	$\Updownarrow$		$\Updownarrow$	
	Code B_1	$\not\equiv$	Code B_2	$\not\equiv \cdots$
$\Updownarrow$	$\Updownarrow$		$\Updownarrow$	
	Code C_1	$\not\equiv$	Code C_2	$\not\equiv \cdots$
	$\Updownarrow$		$\Updownarrow$	
Code-form 2	Code D_1	$\not\equiv$	Code D_2	$\not\equiv \cdots$
	$\Updownarrow$		$\Updownarrow$	
	Code E_1	$\not\equiv$	Code E_2	$\not\equiv \cdots$
$\Updownarrow$	$\Updownarrow$		$\Updownarrow$	

	Code F_1	丰	Code F_2	丰 …
	◇		◇	
	Code G_1	丰	Code G_2	丰 …
	◇		◇	
	Code H_1	丰	Code H_2	丰 …
Code-form 3	◇		◇	
	Code I_1	丰	Code I_2	丰 …
◇	◇		◇	
	Code J_1	丰	Code J_2	丰 …
	◇		◇	
	Code K_1	丰	Code K_2	丰 …
	◇		◇	
⋮	⋮		⋮	⋮

The sign "◇" in the above graph signifies equivalence, not identity. "丰" signifies non-equivalence.

4

A proposition "p" is said to be rejected as false by a basalized system S if "∼p" is recognized as true by S (asserted or deducible).

A basalized system S is said to be inconsistent if there is a proposition which S both recognizes as true and rejects as false.

It can be shown that in an inconsistent system every proposition is at once recognized as true and rejected as false. For if a certain proposition "p" is recognized as true and also rejected as false by the system, this means that both "p" and "∼p" are recognized as true, asserted or deducible. Then any proposition "q" mentionable in the system can be deduced thus: From "∼p" we deduce "∼pvq". From "p" and "∼pvq" we deduce "q". Similarly we can deduce "∼q". It is also evident that in an inconsistent (typified) system there is only one concept in each type, since a proposition of the from "a=b" is always recognized as true (though at the same time rejected as false, as any proposition "a=a" is also rejected).

A basalized system S is said to be inexhaustive if there is a proposition, definable in terms of the concepts in S, which S neither recognizes as true nor rejects as false. We see readily that no inconsistent system is inexhaustive.

A basalized system is said to be exhaustive if it is neither inconsistent nor inexhaustive. When this is the case, the corresponding typified system, primary system and codes, the corresponding base-form, typified system-form, primary system-form and code-forms are all said to be exhaustive.

An exhaustive typified system is said to be finite if the number of primitive non-logical types is finite and if the number of objects (elements) in each primitive non-logical type is also finite. The corresponding systems and system-forms are all said to be finite.

From now on only finite exhaustive systems will be considered in this paper. The two adjectives will be understood.

A typified system having more than one primitive non-logical type can always be reduced to an equivalent typified system having only one primitive non-logical type, which I call the fundamental type. In particular, we may mingle all elements in all the primitive non-logical types of the given system into one (fundamental) type, whose cardinal number is then the sum of the cardinal numbers of those types. (Types are mutually exclusive.) After this is done, there is no difficulty in making adjustments in the mechanism of significance. (There are of course different though equivalent ways of doing this; for example, one might either extend the truth of a certain relation or the falsehood of it over all the relevant types originally foreign to it. Strictly speaking, what results is always a new relation, the old relation being typically inadequate for getting along in the new system. It should be noted that the original types are still to be made mentionable as classes.)

A typified system having one and only one primitive non-logical type is said to be monofundamental. We shall always denote this fundamental type by T_1 and the type of truth-values by T_0. All other types are derivative.

A relational predicate is an object belonging to a type T_0T_1, or $T_0T_1T_1$, or $T_0T_1T_1T_1$, etc. The number of "T_1"s repeated in the designation of the type is called the degree of the relational predicate. In what follows I shall always use the word "relation" for "relational predicate." A relation of degree 1 is called a class. The number of relations, mentionable or not mentionable, which have the degree s is always 2^{n^s}, where n is the number of elements in T_1. A truth-value may be considered as a relation of degree 0. ($2^{n^0}=2$.)

A monofundamental basalized system can always be reduced to an equivalent basalized system (without affecting the fundamental type) in which the only primitive concepts are relations (relational predicates). A proof of this statement is too complicated to be given here.

Finally, it is always possible to reduce these relations to only one primitive relation whose degree is n-1, where n is the number of elements in T_1. Reduction to a degree less than n-1 is not always possible, but that to a higher degree is always possible. In particular, the degree n is convenient for theoretic discussions.

To show in a rough way the truth of our last statements we introduce the notation of ordered "s-ads". An ordered triad will be denoted thus—"⌈abc⌋," which is an abbreviation of "$\breve{R}$(Rabc)." This is of type $T_0.T_0T_1T_1T_1$, or, generally, T_0T, where T is the type of R. We can see that "⌈abc⌋ R" means Rabc. The notation is to be extended to other degrees. Let us now think of the system as given with many primitive relations, R_1, R_2, R_3, etc., which have degrees s_1, s_2, s_3, etc. Then if we denote the elements in T_1 arbitrarily by $a_1, a_2, a_3, \cdots a_n$, and write down all the $\sum n^s$ forms of gR, where g is an s-ad, s being the degree of R, we shall find that some of these are truth and others falsehood.

When the truth or falsehood of these is completely determined, we have the whole system down on paper. We shall now choose an arbitrary n-ad with no repeated elements, say $\lceil a_1 a_2 a_3 \cdots a_n \rfloor$, and we call this D_0; then we put

$$D_0 R_0, \text{ or } R_0 a_1 a_2 a_3 \cdots a_n.$$

We now try one and each of the n!-1 (non-identical) permutations (i.e., substitutions, one-one operations) of elements on our scheme. If a certain permutation does not affect the truth or falsehood of any gR, and if it changes D_0 to the n-ad D_r, we put

$$D_r R_o.$$

If, however, a certain permutation affects the truth or falsehood of some gR, and if it changes D_o to the n-ad D_w, we put

$$\sim\cdot D_w R_o.$$

We thus put down n! statements. Finally, if D_x is an n-ad with some repeated element, we put

$$\sim\cdot D_x R_o.$$

We have then all together n^n statements. We can convince ourselves that these give us an equivalent form of the original scheme, with the many primitive relations reduced to one, R_o, of degree n. If now we omit the last element from each non-repeating n-ad, we still have the same number (n!) of distinct (n-l) ads. Thus we can easily see how the degree of R_o. can be further reduced to n-l. But beyond this point reduction is not always possible. The following scheme, for example, cannot have the degree of its R reduced to 2.

Rabc	∼·Rabd	∼·Raaa
Racd	∼·Racb	∼.Raab
Radb	∼·Radc	Etc., etc., all
Rbad	∼·Rbac	negative for
Rbca	∼·Rbcd	triads with re-

Rbdc	~·Rbda	peating elements.
Rcab	~·Rcad	
Rcbd	~·Rcba	
Rcda	~·Rcdb	
Rdac	~·Rdab	
Rdba	~·Rdbc	
Rdcb	~·Rdca	

From the above discussion it clearly follows that a mono-fundamental typified system is completely determined by the set of permutations (of elements in the fundamental type) which do not affect the scheme. (The scheme, however, is not determined by the set of permutations, since it is already the expression of a basalized system; we can express uniquely a typified system or system-form in a scheme if we use the Shefferian validation variables, and drop the R. In such a scheme the symbols "a", etc., like the validation variables, are "real variables" whose scopes extend beyond the scheme (into discussions about the scheme) but not as far as the scope of the validation variables.) The permutations, since the "product" of any two of them is also found among them as the scheme cannot be affected by both if affected by neither, constitute a permutation group. (The "order" of this group is the number of non-repeating n-ads taking the same validation variable.) Conversely, any permutation group of "degree" n must determine a typified system whose fundamental type consists of the n elements on which the permutations operate. For we can always set up a scheme if we proceed in a way similar to that indicated above; namely, after putting down arbitrarily

$$Ra_1 a_2 \cdots a_n,$$

we perform all the permutation which are in the given group on this n-ad and thus get as many statements as there are permutations in the group. These are affirmative; any other n-ad will be joined negatively with R, and the scheme is complete.

The term "permutation group" is ambiguously used in the Theory of Groups to mean the logical form of a permutation group as well as an actual permutation group. For clarity, I shall prefer the term "permutation group form" or "tropicity" when referring to the form. We have seen that a monofundamental typified system-form is completely determined by a tropicity, and vice versa. I shall therefore use the term "tropicity" ambiguously also for the typified system-form.

A permutation (of elements in T_1) may be considered as an object σ of type $T_1 T_1$, which satisfies the condition $E!(\imath \wp)\{\sqcap \ddot{x}(I: \wp \cdot \sigma x : x)\}$. (We can also think of a permutation as a one-one relation of type $T_0 T_1 T_1$.) Among the n^n objects of type $T_1 T_1$ there are n! which satisfy this condition. (Among the 2^{n^2} objects (relations) of type $T_0 T_1 T_1$ there are n^n one-many (operational) relations, and among these there are n! one-one (permutational) relations.) One of these n! permutations is $\ddot{x}(x)$, the identical permutation which we shall denote by E. The product of a permutation $\wp$ with a permutation σ is defined as $\ddot{x}(\wp \cdot \sigma x)$. We call this $P \wp \sigma$. $P \wp \sigma$ and $P \sigma \wp$ are not in general identical. If $P \wp \sigma = P \sigma \wp$, σ and $\wp$ are "permutable" with each other.

A permutation group is a set of permutations such that the product of any two permutations is a permutation in the set. The following graphs will be found helpful:—

Fig. 0

Fig. 1

Fig. 2

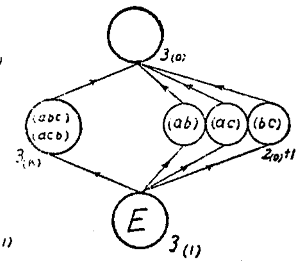


Fig. 3

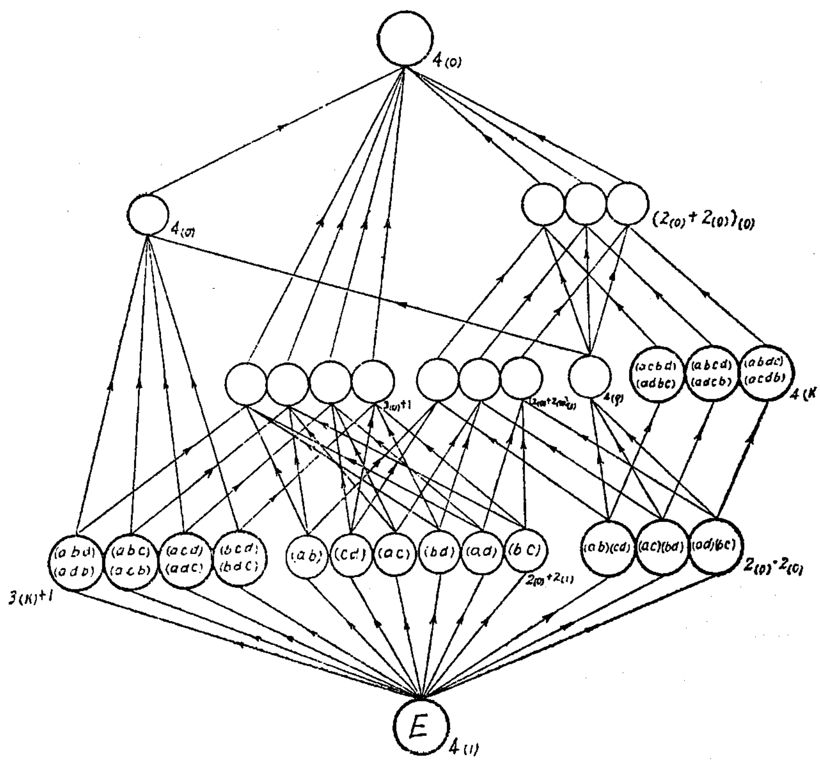


Fig. 4

The series of graphs may continue indefinitely. The circles in the graphs represent permutation groups of degrees o, 1, 2, 3, 4 in Fig. o, 1, 2, 3, 4 respectively. The group represented by any circle C consists of all the permutations written within circles which tend to fly upward to C (immediately or mediately) together with the permutations (if any) written within C.

The graphs have two interpretations:—

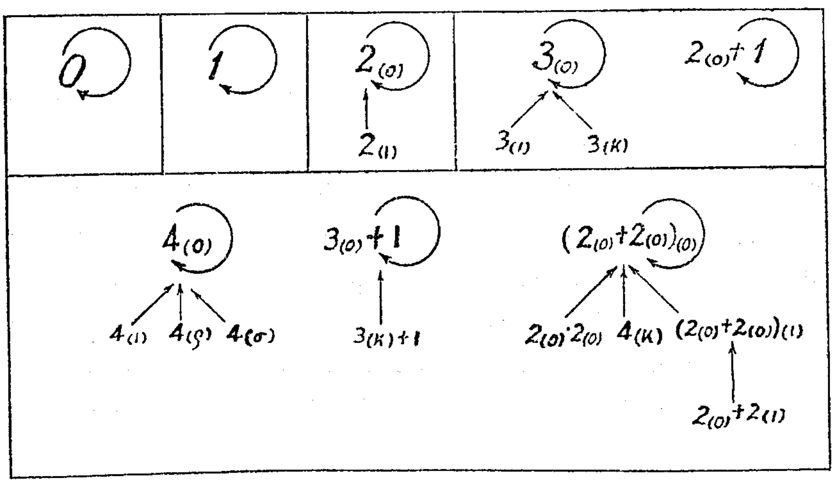
(1) We may consider each graph as representing a mono-fundamental typified system (or system-form) which is of the kind that all objects in the system are mentionable as concepts in the system. (It is clear that such a system-form is completely determined by the cardinal number of elements.) Each circle then represents an infinite set of objects (of various types) which are such that any one of them is definable in terms of any other with the aid of logical concepts. Sets which have the same logical property (form) are represented by circles touching one another. If a circle A tends to fly upward (immediately or mediately) to a circle B, this means that any object in the set represented by B is definable in terms of any object in the set represented by A (with the aid of logical concepts), but not inversely. For any number of circles S in any one graph there is always a circle C in the same graph, either distinct from all or identical with one of S, such that C tends to fly upward to each of the circles S with which it is not identical, and no other circle has this same property unless it tends to fly upward to C. Any object in the set represented by C will be definable in terms of a number of objects in the sets represented by the circles S, if these objects are such that at least one comes from each set. C is called the basic circle of the circles S. If the objects q, u, w, ⋯ are in the sets represented by the circles Q, U, W, ⋯ (not necessarily all distinct), and if C is the basic circle of Q, U, W, ⋯, then if D is a circle such that C tends to fly upward to it, any object in the set represented by D is definable in terms of q, u, w, ⋯ with the aid of logical concepts. Thus there is a complete picturing of relative definability. (If, for example, $R' = \hat{x}(Rxx)$, R' is definable in terms of R, but usually not inversely. The circle representing

the set containing R will therefore, in the usual case, tend to fly upward (that is to say, there will be an arrow going upward from the circle, or a series of arrows  starting from the circle) to the circle representing the set containing R'.)

(2) Again, any circle may be considered as representing the (monofundamental) typified system in which all and only the objects in the set represented by the circle together with all objects definable in terms of them are mentionable. We are not to think that the permutations in the group represented by a circle C are always among objects in the set represented by C; this is frequently not the case. It would be an utter confusion if one supposes that any typified system can be basalized by making the permutations (or a part of them) which determine the system (in the way already described) primitive concepts; in fact, these permutations are not always mentionable in the system. However, it sometimes happens that the typified system initiated by these primitive concepts is identical with the system determined. Such is the case, for example, with a system having the form $3_{(K)}$. But the permutations in a group of the form $3_{(O)}$ initiate a system of the form $3_{(I)}$, and those in a group of the form $3_{(I)}$ initiate a system of the form $3_{(O)}$. Again, while $2_{(O)}$ initiates itself, i. e., $2_{(O)}$, also $2_{(I)}$ initiates $2_{(O)}$. Thus there are systems which cannot be so basalized that all the primitive concepts are permutations (one-one operations); such, for example, is a system of the form $2_{(I)}$. This is a point which we are to bear in mind in order to avoid a possible intuitive confusion.

When we have a monofundamental typified system S, we can always build another monofundamental typified system Q, whose fundamental type is the same as that of S, and whose concepts are all the concepts definable in terms of "S" (not concepts *in* S) with the aid of logical concepts. Q is called the transcendental system of S, and is not usually identical or equivalent to S. The tropicity which is the form of Q is called the transcendental tropicity of the tropicity which is the form of S. Thus the transcendental tropicity of $3_{(k)}$ is $3_{(o)}$, that of $3_{(I)}$ is also $3_{(o)}$, and that of $3_{(o)}$ itself is $3_{(o)}$. But that of $2_{(o)} + 1$ is $2_{(o)} + 1$ itself.

We now represent these facts by the following "genealogies" :—



Thus the tropicities group themselves in families. Each family has a tropicity which is its own transcendental tropicity; we call such a final tropicity. Tropicities which have the same transcendental tropicity form a generation; the transcendental tropicity will be called its inaugurator. If the inaugurator of a generation is a final tropicity, the inaugurator is itself in the generation, and the latter is called a primordial generation; as, for example, the generation consisting of $(2_{(o)} + 2_{(o)})_{(o)}$, $2_{(o)}.2_{(o)}$, $4_{(k)}$ and $(2_{(o)} + 2_{(o)})_{(I)}$, having its own member $(2_{(o)}+2_{(o)})_{(o)}$ as inaugurator. But if the inaugurator of a generation is not a final tropicity, the inaugurator is outside of the generation; thus, the generation consisting of $2_{(o)}+2_{(I)}$ alone has $(2_{(o)}+2_{(o)})_{(I)}$ as inaugurator. A family is composed of mutually exclusive generations, joined by ties of inauguratorship.

It would be an utter mistake to suppose that tropicities of the same family or of the same generation are all equivalent (i. e., to suppose that they come under the same primary system-form). Thus $3_{(k)}$ is certainly not equivalent to $3_{(o)}$; for in a system of the form $3_{(k)}$ we have the mentionable one-one relation R whose scheme is Rab, Rbc, Rca,~Rba,~Rcb,~Rac, etc., and which is not definable in the system of the form $3_{(o)}$. In general, whenever two systems having different tropicities are equivalent, the fundamental type of the one system is not translated into the fundamental type of the other; in other words, two systems having the same fundamental type (mutually translatable types) but different tropicities are never equivalent. These remarks serve to explain why I do not accept Dr. Sheffer's "straticities" as criteria for equivalence.

The number of circles in a graph which touch one another and which together represent a tropicity is called the coefficient of the tropicity. The number of permutations (including the identical permutation) in a group having a tropicity H is called the order or the indifference measure of H. The number of elements in the fundamental type of a system whose form is the tropicity H is called the degree of H. If n is the degree of H, and i the order of H, we define the potential of H as n!/i. Let p be the potential, and we have

$$i \cdot p = n!$$

If the transcendental tropicity of H is H', the coefficient of H is equal to the potential of H'. I call this the Coefficient Theorem. If H is a final tropicity, its potential is equal to its coefficient. If not, the potential of H is a multiple of its coefficient. If c denotes coefficient, the rate r of a tropicity is defined as p/c. We have

$$c \cdot r = p,$$

$$i \cdot c \cdot r = n!$$

Following is a list of the potentials, coefficients, etc., of tropicities of degrees not higher than 4:—

Degree n	Tropicity H	Order i	Potential p	Coefficient c	Rate r	Transcendental Tropicity of H
0	0	1	1	1	1	0
1	1	1	1	1	1	1
2	$2_{(o)}$	2	1	1	1	$2_{(o)}$
	$2_{(I)}$	1	2	1	2	
3	$3_{(o)}$	6	1	1	1	$3_{(o)}$
	$3_{(k)}$	3	2	1	2	
	$3_{(I)}$	1	6	1	6	
	$2_{(o)}+1$	2	3	3	1	$2_{(o)}+1$
4	$4_{(o)}$	24	1	1	1	$4_{(o)}$
	$4_{(\sigma)}$	12	2	1	2	
	$4_{(\varrho)}$	4	6	1	6	
	$4_{(I)}$	1	24	1	24	
	$3_{(o)}+1$	6	4	4	1	$3_{(o)}+1$
	$3_{(k)}+1$	3	8	4	2	
	$(2_{(o)} + 2_{(o)})_{(o)}$	8	3	3	1	$(2_{(o)} + 2_{(o)})_{(o)}$
	$4_{(k)}$	4	6	3	2	
	$2_{(o)} \cdot 2_{(o)}$	2	12	3	4	
	$(2_{(o)} + 2_{(o)}`_{(I)}$	4	6	3	2	
	$2_{(o)} + 2_{(I)}$	2	12	6	2	$(2_{(o)} + 2_{(o)})_{(I)}$

It is clear that a tropicity whose rate is l is a final tropicity, and conversely. The potential of a tropicity is extremely important from the logical point of view. It is equal to the number of objects having the same logical form (property), when any one of these objects is in a set represented by one of those circles which together represent the tropicity. (This logical form is to be defined only in terms of the fundamental type and the logical concepts, without using the primitive non-logical concepts of the system. The case in which we admit these latter gives rise to the "derivative potential", which I shall not discuss in this paper.)

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We shall now give some geometrical examples of our tropicities. In each case we give only the elements of the fundamental type; the concepts of the system will be all those which are definable in terms of geometrical concepts and the fundamental type. It is to be noted that "one inch long" or "three inches long" is not a geometrical concept. But "one degree" (angle), "ninety degrees", "three times as long", etc., are geometrical concepts. There is one concept which in some cases we shall add to geometrical concepts; namely, the concept Clc (clockwise). "Clc p q r s" means "the four points p, q, r and s do not all lie in the same plane, and from the point of view of p a circular motion from q to r to s and back to q is clockwise." It facilitates our imagination to think of a permutation which does not affect a certain system as one resulting from turning (and translating, if necessary) our figure until it coincides with itself again. Thus a square may be turned in seven ways until it coincides with itself again, each way resulting in a different arrangement of the vertices. There are thus eight permutations (including the identical permutation) in the tropicity. In the case of a solid we cannot actually turn it through a fourth dimension, and thus we shall have recourse to mirroring as well as turning. Wherever we shall make a point that we add Clc to our concepts (we write "Add Clc"), this will mean that mirroring is not allowed in the case under consideration. On the other hand, wherever we shall make a point that we do not add Clc to our concepts (we write "Add no Clc"), this will mean that turning cannot exhaust the permutations, and we must have recourse to mirroring also. When there is no mention of either condition, the implication is that addition or non-addition of Clc to our concepts does not affect our system in the least, and that therefore we need not use mirroring, since mirroring does not give any permutation not already given by turning. Theoretically speaking, uniform expansion or contraction will also be a legitimate operation. But since in all our illustrations the figures are of finite size, it is clear that expansion will not help us in the least, although it is allowable.

$2_{(0)}$—Two distinct points.

$2_{(1)}$—Two (distinct) concentric spheres.

$3_{(0)}$—The three vertices of an equilateral triangle.

$3_{(k)}$—The three edges of a regular tetrahedron adjacent to one of its vertices. Add Clc.

$2_{(0)}+1$—The three vertices of an isosceles, but not equilateral, triangle.

$3_{(1)}$—The three vertices of a scalene triangle.

$4_{(0)}$—The four vertices of a regular tetrahedron. Add no Clc.

$(2_{(0)}+2_{(0)})_{(0)}$—The four vertices of a square.

$4_{(k)}$—The four edges of a regular square pyramid adjacent to the vertex. Add Clc.

$(2_{(0)}+2_{(0)})_{(1)}$—The four vertices of a rhombus which is not a square.

$2_{(0)}\cdot 2_{(0)}$—The four vertices of an isosceles trapezoid not a rectangle.

$3_{(0)}+1$—The three vertices and the center of an equilateral triangle.

$3_{(k)}+1$—One vertex and the three adjacent edges of a regular tetrahedron. Add Clc.

$2_{(0)}+2_{(1)}$—The three vertices of an isosceles triangle and the mid-point of the base.

$4(_{\sigma})$—The four vertices of a regular tetrahedron. Add Clc.

$4(\varrho)$—The four vertices of a rectangle not a square.

$4_{(1)}$—Four (distinct) concentric spheres.

NOTE.—This paper is left unfinished. The writer finds himself not in a position to continue it. The extensional postulates, together with the principle (not given in the paper) that two mentionable classes having a one-one correlator which is also mentionable and known as one-one are to be considered as mutually translatable (it is to be remarked that some such principle is necessary in order to establish the equivalence, for example, of the Huntingtonian tropicity with the tropicity resulting from making all Euclidean points the elements) naturally lead to the consequence that a primary system-form is completely determined by a "group" or rather "group form" (not "permutation group form"), and vice versa. With primary systems which contain no objects which are not mentionable there would be no difference whatever in logical form. Since a system which does not satisfy this condition fails to do so, just as is the case of inexhaustiveness, only by reason of subjective indetermination, the conclusion is that there could be no differentiation in extensional objective order. The inadequacy of the extensional point of view for dealing with problems of systematic order is thus evident, and a more fundamental consideration seems necessary.

Additional Note.—The present article was written several years ago. The "National Relativity" referred to in the paper was an unpublished former article by Professor Sheffer, where the term "tropicity", which he adopted later on, was not yet introduced. The method which I used in the paper is not from Sheffer, so that I am solely responsible for any mistake which might occur.