國立清華大學 101 學年度碩士班考試入學試題

系所班組別：數學系純粹數學組

考試科目（代碼）：高等微積分（0101）

　　*請在【答案卷】作答

1. (20 points) Let E be a set in $\mathbb{R}^2$ and $\pi : \mathbb{R}^2 \to \mathbb{R}$ is the projection map $\pi(x, y) = x$.

   (a) If E is compact, should $\pi(E)$ be compact? Explain! (6 points)

   (b) If E is open, should $\pi(E)$ be open? Explain! (7 points)

   (c) If E is closed, should $\pi(E)$ be closed? Explain! (7 points)

2. (16 points) Evaluate

   (a) $\lim_{n\to\infty}((1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{n}{n}))^{1/n}$. (8 points)

   (b) $\int_{-1}^{2} |x| d|x|$. (8 points)

3. (12 points) Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that f is Riemann integrable on $[0, 1]$. What is $\int_0^1 f(x)dx$?

4. (10 points) Show that the function $f(x) = \sum_{n=1}^{\infty} \frac{x^2}{x^2+n^2}$ is continuous on $\mathbb{R}$.

5. (10 points) Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\mathbf{x}) = \|A\mathbf{x}\|$, where A is a nonsingular $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. Check at what $\mathbf{x}$ is f differentiable and find $Df(\mathbf{x})$.

6. (10 points) Let $F(x)$ be defined by

$$F(x) \equiv \int_0^x \left(\int_t^x \sqrt{1+s^3}ds \right) dt.$$

   Explain why F is differentiable at each $x \in (0, \infty)$ and find $F'(x)$.

7. (10 points) Suppose U is a non-empty open set in $\mathbb{R}^2$ and $f : U \to \mathbb{R}^2$ is continuously differentiable with its Jacobian $J(f)(\mathbf{x}) \neq 0$ on U. Show that
$$\lim_{r\to 0^+} \frac{area(f(B_r(\mathbf{x}))}{area(B_r(\mathbf{x}))} = |J(f)(\mathbf{x})|,$$
for every $\mathbf{x} \in U$. ($B_r(\mathbf{x})$ is the disc of radius r centered at $\mathbf{x}$.)

8. (12 points) Consider the vector field $\mathbf{v} = (2xe^y - y\sin x,\ x^2e^y + \cos x + 2y)$, and let C be the semi-circle $\{(x,y) : x \geq 0, x^2+y^2 = 1\}$ oriented counterclockwise.

   (a) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ so that $\nabla f = \mathbf{v}$. (6 points)

   (b) Evaluate the line integral
   $$\int_C \mathbf{v} \cdot d\mathbf{r}.$$
   (6 points)