

# 國立清華大學 102 學年度碩士班考試入學試題

系所班組別：計量財務金融學系

考試科目（代碼）：微積分(4504)

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\*請在【答案卷、卡】作答

## Calculus

1. (10 pts) Let

$$F(x) = \int_0^{\pi} |x - t| \cos t \, dt.$$

Find  $F'(x)$  for  $0 < x < \pi$ .

2. (10 pts) Find the area of the region

$$\Omega = \{(x, y) / 2x^2 + 2xy + 5y^2 \leq 1\}.$$

Hint:  $2x^2 + 2xy + 5y^2 = (x + 2y)^2 + (x - y)^2$ .

3. (12 pts) Maximize the function

$$f(x, y, z) = 2^x 3^y 5^z$$

subject to the constraint  $x^2 + y^2 + z^2 = 1$ .

4. (12 pts) If  $0 < x < \pi/2$ , prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

5. (16 pts) Let  $f(x, y) = 4x^3 - 12x^2 + y^2 - 36x + 6$ .

- Find the critical point(s) of  $f$ .
- Classify the nature of the critical point(s).
- Find the relative extremum of  $f$ , if it exists.

6. (20 pts) Let  $a_n$  be defined by

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n.$$

- Show that  $\{a_n\}$  is a decreasing sequence of positive numbers.
- Does  $\lim_{n \rightarrow \infty} a_n$  exist? Why?
- Show that the alternating series  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$  converges to  $\log 2$ .

7. (20 pts) Let  $y(t)$  be the size of a quantity at time  $t$ . Suppose that  $y(t) \leq Y$  for all  $t \geq 0$ , and that  $y$  satisfies the differential equation

$$\begin{cases} dy/dt = ky(Y - \log y) \\ y(0) = y_0 \end{cases},$$

where  $Y$ ,  $k$  and  $y_0$  are positive constants.

- Solve this differential equation.
- Find the time  $t$  at which the growth rate  $y'(t)$  is increasing most rapidly.