High temperature susceptibility series for the spin-1/2 anisotropic Heisenberg model

1973 J. Phys. C: Solid State Phys. 6 2713

(http://iopscience.iop.org/0022-3719/6/17/012)

J. Phys. C: Solid State Phys., Vol. 6, 1973. Printed in Great Britain. © 1973

High temperature susceptibility series for the spin-$\frac{1}{2}$ anisotropic Heisenberg model†

D C Jou and H H Chen
Institute of Physics, National Tsing Hua University, Hsinchu, Taiwan, Republic of China

Received 24 April 1973, in final form 13 June 1973

Abstract. The exact high temperature susceptibility series for the spin-$\frac{1}{2}$ uniaxially anisotropic Heisenberg model is derived through order T^{-7} for various lattices and anisotropies. The series is extrapolated by the Padé approximant method to give the critical parameters for a ferromagnet having spontaneous magnetization parallel to the symmetry axis of the system. When the anisotropy changes from the Ising limit to the isotropic Heisenberg limit the critical temperature of the system decreases, slowly in the Ising limit and rapidly in the isotropic limit. Our results are consistent with the hypothesis that the critical index remains constant as the anisotropy varies from the Ising limit toward the isotropic limit, and changes discontinuously when the system becomes isotropic.

1. Introduction

Wood and Dalton (1972) have recently calculated the high temperature series expansions for the uniaxially anisotropic Heisenberg model described by the hamiltonian

$$\mathscr{H} = -2J \sum_{\langle ij \rangle} [S_{iz}S_{jz} + \eta(S_{ix}S_{jx} + S_{iy}S_{jy})], \tag{1}$$

where the summation is taken over all pairs of nearest neighbour ions in a regular crystal lattice of N sites, S_{ix}, S_{iy} and S_{iz} are the cartesian components of the spin operator $\boldsymbol{S}_i$ of the ion located at the lattice site labelled i, J is the exchange constant and η an anisotropy parameter. When $\eta = 0$ and 1, the above hamiltonian reduces to the Ising model and the isotropic Heisenberg model, respectively. For $\eta \gg 1$, the hamiltonian reduces to the XY model. It is known that below the critical temperature the system is in a ferromagnetic phase if the constants J and η are positive. For $\eta < 1$ the spontaneous magnetization of the system orders along the z axis; and for $\eta > 1$ the magnetization lies in the xy plane. The determination of the critical parameters of the system has been a subject of theoretical interest for many years. The critical temperature T_c and the critical index γ are defined through the zero-field susceptibility by

$$\chi \sim (T - T_c)^{-\gamma} \qquad \text{for } T \to T_c^+. \tag{2}$$

From the molecular field approximation (Smart 1966) it is found that $\gamma = 1$ for all values of η, and $T_c = 2JqS(S+1)/3k$ for $0 \leqslant \eta \leqslant 1$ and $T_c = 2J\eta qS(S+1)/3k$ for $\eta > 1$. Here q is the coordination number of the crystal lattice, and k the Boltzmann con-

† Work supported by the National Science Council of the Republic of China.

stant. The Oguchi approximation (H H Chen, unpublished) predicts that $\gamma = 1$ for all values of η and T_c decreases as η increases from 0 to 1. The result that $\gamma = 1$ for all values of η is a consequence of the effective field approximation (Smart 1966). Dalton and Wood (1967) have determined the critical parameters by using the Green function technique. It is found that in the random phase approximation the critical temperature decreases as η increases. The susceptibility index remains constant ($\gamma = 1$) for $0 \leqslant \eta < 1$, and changes discontinuously ($\gamma = 2$) at $\eta = 1$. The discontinuity of γ at $\eta = 1$, where the system becomes isotropic, is an example of the so called 'universality hypothesis' (Griffiths 1970).

The results of the above-mentioned approximations are clearly inaccurate for $\eta = 0$ as compared to the exact values of T_c $(=J/\ln(1+\sqrt{2})k)$ and γ $(=\frac{7}{4})$ for the two-dimensional Ising model (Yang 1952, Fisher 1959). The most reliable approximation for estimating the critical parameters is believed to be the extrapolation of the exact high temperature series expansions for the zero-field susceptibility. For the uniaxially anisotropic hamiltonian the susceptibility is a diagonalized cartesian tensor with elements χ_{xx}, χ_{yy} $(=\chi_{xx})$ and χ_{zz}. At T_c, χ_{zz} diverges if $\eta < 1$; and χ_{xx} diverges if $\eta > 1$. For the Ising model ($\eta = 0$) and the isotropic spin-$\frac{1}{2}$ Heisenberg model ($\eta = 1$) considerable long series expansions for χ_{zz} have been derived and used to estimate the critical parameters (Domb and Sykes 1962, Baker *et al* 1967). For a general value of η Dalton and Wood (1967) have estimated the critical parameters for spin-$\frac{1}{2}$ systems from the high temperature susceptibility series derived only to order T^{-5}. Wood and Dalton (1972) have also calculated the series χ_{zz} through order T^{-6} for general spin and anisotropy. Analysis of the series, however, has not been reported. Obokata *et al* (1967) obtained terms through order T^{-7} in the series for spin-$\frac{1}{2}$ and for the linear chain, simple quadratic and simple cubic lattices. A high temperature series expansion for χ_{xx} is difficult to derive and has not been obtained because the ordering parameter $\mathscr{S}_x = \Sigma_i S_{ix}$ does not commute with the anisotropic hamiltonian. The critical parameters T_c and γ were estimated from the high temperature series for the mean-square fluctuation of $\mathscr{S}_x$ (Betts *et al* 1970, Jou and Chen 1973). In the classical limit, $S \to \infty$, the non-commutivity of the spin operators can be neglected, and the high temperature series expansions are easier to obtain. Series expansions for χ_{zz} and χ_{xx} were calculated by Jasnow and Wortis (1968). In this limit their results supported the universality hypothesis.

Here we extend the high temperature series expansions for χ_{zz} for the spin-$\frac{1}{2}$ anisotropic Heisenberg model to order T^{-7} for arbitrary lattice and general value of η. Our series is calculated by using the cluster expansion method (Domb 1960) which is discussed in the following section. The critical temperature and the critical index are estimated from the series by means of Padé approximant method (Baker 1961, 1963). Our results are consistent with the hypothesis that the critical index γ remains constant for $0 \leqslant \eta < 1$ and changes discontinuously at $\eta = 1$.

2. Cluster expansion method

Derivations of the high temperature series expansions for various hamiltonians are based either on the diagrammatic method developed by Rushbrooke and Wood (1958) or on the cluster expansion method introduced by Domb (1960). In the diagrammatic method a great number of diagrams contribute to the series coefficients and a lot of hand calculations are involved. In the cluster expansion method, on the other hand, a smaller number of clusters are involved and most of the calculations can be handled by electronic computers. The only disadvantage of the cluster expansion method is that the spin value generally can not be treated as a variable. The high temperature series expansions have to be derived for a fixed value of the spin (Domb and Wood 1965, Dalton and Wood 1967, Baker *et al* 1967). And for $S > \frac{1}{2}$ the cluster expansion method rapidly becomes impracticable because of the large dimensions of the matrix representations (Allan and Betts 1967, Chen and Levy 1973). The only model where the spin value has been treated as a variable in the cluster expansion method is the exchange interaction model (Chen and Joseph 1972). Methods for treating the spin value as a variable for other models have not been developed. The high temperature susceptibility series for the spin-S anisotropic Heisenberg model to order T^{-6} (Wood and Dalton 1972) were calculated by the diagrammatic method. In this paper we extended the series to order T^{-7} for spin-$\frac{1}{2}$ systems by using the cluster expansion method.

In the cluster expansion method the zero-field susceptibility per ion of a regular crystal lattice G is written in the form

$$\chi_{zz}(G, T, \eta) = \sum_{i=1}^{\infty} (g_i; G) f_i(T, \eta), \tag{3}$$

where i sums over all clusters and $(g_i; G)$ is the lattice constant of the cluster g_i in the lattice G. The quantities $f_i(T, \eta)$ depend on the temperature and the form of interactions in the lattice, and are given recursively by

$$f_i(T, \eta) = \chi_{zz}(g_i, T, \eta) - \sum_{j=1}^{i-1} (g_j; g_i) f_j(T, \eta). \tag{4}$$

In order to calculate the high temperature series for $\chi_{zz}(G, T, \eta)$ to order T^{-l} one has to calculate $\chi_{zz}(g_i, T, \eta)$ for all clusters containing up to l pairs of interactions. For example, one has to consider 52 clusters to obtain the susceptibility series to order T^{-6}, and 131 clusters to order T^{-7}. Note that Wood and Dalton (1972) considered more than 300 diagrams to obtain the susceptibility series to order T^{-6}.

The calculation of $\chi_{zz}(g_i, T, \eta)$ is straightforward. For a cluster containing n ions, the dimension of the matrices $\mathscr{H}$ and $\mathscr{S}_z$ $(=\Sigma_i S_{iz})$ is $(2S + 1)^n \times (2S + 1)^n$, or $2^n \times 2^n$. Since the hamiltonian commutes with $\mathscr{S}_z$, the hamiltonian does not have matrix elements between states having different eigenvalues of $\mathscr{S}_z$, ie the hamiltonian is a direct sum of submatrices associated with different eigenvalues of $\mathscr{S}_z$. The dimension of the submatrix associated with $\mathscr{S}_z = m$ is just $n!/[(n/2 + m)!(n/2 - m)!]$. For a cluster with seven ions, for instance, the dimension of matrices is 128×128, while the largest size of submatrices is 35×35 (associated with $\mathscr{S}_z = \pm\frac{1}{2}$). The traces of the submatrices of $\mathscr{H}^n$ and $\mathscr{H}^n\mathscr{S}_z^2$ for $n \leqslant 7$, and hence χ_{zz}, are calculated by computer.

3. Susceptibility series

It is convenient to express the zero-field high temperature susceptibility series in the form

$$\chi_{zz} = (C/T)\left[1 + \sum_{n=1}^{\infty} b_n(\eta)(J/2kT)^n/n!\right], \tag{5}$$

where $C(=Ng^2\mu^2/4k)$ is the Curie constant. The coefficients $b_1(\eta)$ through $b_5(\eta)$ have been found by Dalton and Wood (1967). The coefficients $b_6(\eta)$ and $b_7(\eta)$ are given below:

$$
\begin{aligned}
b_6(\eta) = {} & 720q\sigma^5 - 3600q\sigma^4\eta^2 + 240q\sigma^3(21\eta^4 + 12\eta^2 - 4) - 16q\sigma^2(89\eta^6 \\
& + 294\eta^4 - 129\eta^2) - 16q\sigma(64\eta^6 + 216\eta^4 + 60\eta^2 - 17) - 16q(17\eta^6 \\
& + 30\eta^4 + 30\eta^2) - 5760p_3\sigma^3(2\eta^3 + 3) + 1440p_3\sigma^2(16\eta^5 + 8\eta^3 + 45\eta^2) \\
& - 576p_3\sigma(4\eta^5 + 65\eta^4 - 19\eta^3 + 58\eta^2 - 35) + 96p_3(73\eta^6 - 96\eta^5 + 120\eta^4 \\
& + 86\eta^3 - 126\eta^2 + 45) - 5760p_4\sigma^2(2\eta^4 + 3) + 384p_4\sigma(34\eta^6 + 44\eta^4 \\
& + 130\eta^2) + 1920p_4(4\eta^6 - 7\eta^4 - 4\eta^2 + 7) - 4800p_5\sigma(2\eta^5 + 3) \\
& + 480p_5(12\eta^5 + 35\eta^2) + 1920p_{5a}\sigma(\eta^4 + 2\eta^3 + 6) + 192p_{5a}(45\eta^6 \\
& - 12\eta^5 + 146\eta^3 - 122\eta^2 + 105) - 2880p_6(2\eta^6 + 3) + 576p_{6a}(7\eta^4 + 10) \\
& + 192p_{6b}(6\eta^5 + 7\eta^4 + 5\eta^3 + 30) + 384p_{6c}(12\eta^6 + 50\eta^3 + 15) + 3072p_{6d}\eta^3, \\
b_7(\eta) = {} & 5040q\sigma^6 - 30240q\sigma^5\eta^2 + 1680q\sigma^4(34\eta^4 + 16\eta^2 - 5) - 224q\sigma^3(149\eta^6 \\
& + 339\eta^4 - 99) + 112q\sigma^2(134\eta^6 - 196\eta^4 - 136\eta^2 + 33) + 448q\sigma(51\eta^6 \\
& + 37\eta^4 - 17\eta^2) + 16q(420\eta^6 + 420\eta^4 - 17) - 50400p_3\sigma^4(2\eta^3 + 3) \\
& + 20160p_3\sigma^3(15\eta^5 + 6\eta^3 + 37\eta^2) - 1344p_3\sigma^2(109\eta^7 + 248\eta^5 + 690\eta^4 \\
& - 70\eta^3 + 426\eta^2 - 180) - 1344p_3\sigma(100\eta^7 - 168\eta^6 + 257\eta^5 - 447\eta^4 - 40\eta^3 \\
& + 322\eta^2 - 45) - 1344p_3(22\eta^7 + 75\eta^6 + 65\eta^5 + 24\eta^4 + 60\eta^3 + 20\eta^2 + 54) \\
& - 53760p_4\sigma^3(2\eta^4 + 3) + 5376p_4\sigma^2(44\eta^6 + 39\eta^4 + 130\eta^2) \\
& - 1792p_4\sigma(88\eta^6 + 357\eta^4 + 192\eta^2 - 120) - 3584p_4(3\eta^6 - 49\eta^4 + 48\eta^2) \\
& - 50400p_5\sigma^2(2\eta^5 + 3) + 2240p_5\sigma(56\eta^7 + 77\eta^5 + 225\eta^2) + 13440p_5(6\eta^7 \\
& - 9\eta^4 - 5\eta^2 + 10) + 20160p_{5a}\sigma^2(\eta^4 + 2\eta^3 + 6) + 896p_{5a}\sigma(106\eta^6 - 89\eta^5 \\
& - 9\eta^4 + 426\eta^3 - 606\eta^2 + 315) + 448p_{5a}(434\eta^7 - 104\eta^6 - 674\eta^5 + 988\eta^4 \\
& + 60\eta^3 - 552\eta^2 + 255) - 40320p_6\sigma(2\eta^6 + 3) + 2688p_6(23\eta^6 + 60\eta^2) \\
& + 8064p_{6a}\sigma(7\eta^4 + 10) - 2688p_{6a}(17\eta^6 + 11\eta^4 + 60\eta^2 - 30) \\
& + 2688p_{6b}\sigma(6\eta^5 + 7\eta^4 + 5\eta^3 + 30) + 224p_{6b}(316\eta^7 - 44\eta^6 - 114\eta^5 + 298\eta^4 \\
& + 528\eta^3 - 816\eta^2 + 675) + 5376p_{6c}\sigma(12\eta^6 + 50\eta^3 + 15) - 1792p_{6c}(7\eta^6 \\
& + 123\eta^5 + 24\eta^3 + 102\eta^2 - 45) + 43008p_{6d}\sigma\eta^3 + 10752p_{6d}(3\eta^6 - 7\eta^5 \\
& + 5\eta^4 - 4\eta^3 + 12\eta^2) - 23520p_7(2\eta^7 + 3) + 896p_{7a}(11\eta^6 + 21\eta^4 + 45) \\
& + 448p_{7b}(17\eta^6 + 20\eta^5 + 15\eta^3 + 90) + 448p_{7c}(34\eta^6 + 10\eta^5 + 24\eta^4 + 75\eta^3 \\
& + 90) + 4480p_{7d}(4\eta^6 + 12\eta^3 + 9) + 2688p_{7e}(14\eta^7 + 27\eta^4 + 30\eta^3 + 15) \\
& + 448p_{7f}(40\eta^5 + 21\eta^4 + 90) + 4032p_{7g}(58\eta^7 + 66\eta^4 + 76\eta^3 + 55) \\
& + 448p_{7h}(6\eta^5 - 7\eta^4 + 24\eta^3).
\end{aligned}
$$

Here $\sigma + 1 = q$, and p_{nx} are the lattice constants in Domb's notation (Domb 1960).

For $\eta = 0$ and 1 our results reduce exactly to those of the Ising model (Domb and Sykes 1962) and the isotropic Heisenberg model (Baker *et al* 1967), respectively. For a general value of η our results agree with those of Wood and Dalton (1972)† and Obokata *et al* (1967).

4. Critical parameters

As mentioned in the Introduction, the zero-field susceptibility series χ_{zz} can be used to estimate the critical parameters for systems ordering along the z axis ferromagnetically. It is clear that the present series can be used to extrapolate the critical parameters for $J > 0$ and $0 \leqslant \eta \leqslant 1$. Before estimating these parameters numerically, two important results can be found by inspecting the η-dependence of the series coefficients:

$$b_n(\eta) = b_{n0} + b_{n1}\eta + b_{n2}\eta^2 + \ldots + b_{nn}\eta^n. \tag{6}$$

From our results we find that the constants b_{n1} vanish for all lattices and b_{nm} vanish for loose-packed lattices and for m odd. This implies that for any lattice

$$\lim_{\eta \to 0} \mathrm{d}T_c(\eta)/\mathrm{d}\eta = 0, \tag{7}$$

and for a loose-packed lattice

$$T_c(-\eta) = T_c(\eta) \qquad \gamma(-\eta) = \gamma(\eta). \tag{8}$$

Equations (8) are true as long as η is not too large that the spontaneous magnetization of the system orders in the xy plane.

To determine the critical temperature and the critical index from the susceptibility series by Padé approximants, three methods are commonly used:

(i) Choosing γ, $K_c = J/kT_c$ is presented by appropriate poles of the Padé approximants to $(\chi)^{1/\gamma}$.

(ii) Choosing K_c, γ is obtained by evaluating Padé approximants to $(K - K_c)$ $(\mathrm{d}\ln\chi/\mathrm{d}K)$ at $K = K_c$.

(iii) For a Padé approximant to $\mathrm{d}\ln\chi/\mathrm{d}K$ the appropriate pole gives K_c and the residue at this pole gives $-\gamma$.

Among the three methods only method (iii) predicts the parameters K_c and γ simultaneously. In this method, however, the various approximants are rather diverse and the estimates are less accurate. In methods (i) and (ii) the various approximants are very regular, but the estimates of K_c and γ depend on the preknowledge of each other. To determine K_c and γ simultaneously and accurately we have combined methods (i) and (ii) by plotting in the K_c–γ plane the various approximants of K_c as functions of γ (method i) and the various approximants of γ as functions of K_c (method ii). The proper values of K_c and γ are obtained from these curves in the region in which the various approximants coalesce. By this method our estimates for the critical parameters are not influenced by the preknowledge of the critical parameters previously obtained by other authors. We have also estimated the critical parameters by the ratio method. The uncertainties in the estimates of the ratio method are larger but are not inconsistent with those obtained by the Padé approximant method.

† There are two errors in $b_6(\eta)$ of Wood and Dalton (1972). The coefficients of $p_{5a}X^4\eta^5$ and $p_{5a}X^4\eta^6$ should read $-216760320p_{5a}X^4\eta^5 - 92897280p_{5a}X^4\eta^6$.

Table 1. Critical temperatures kT_c/J for several crystal lattices†

Anisotropy η	FCC lattice	BCC lattice	SC lattice	Plane triangular	Simple quadratic
0	4·90	3·18	2·26	1·82 ± 0·01	1·13 ± 0·01
0·1	4·89	3·17	2·26	1·82 ± 0·01	1·13 ± 0·01
0·2	4·88	3·16	2·25	1·81 ± 0·01	1·12 ± 0·01
0·3	4·86	3·14	2·23	1·79 ± 0·01	1·11 ± 0·01
0·4	4·83	3·11	2·20	1·75 ± 0·02	1·09 ± 0·01
0·5	4·78	3·08	2·17	1·70 ± 0·03	1·05 ± 0·02
0·6	4·71	3·03	2·12	1·62 ± 0·04	0·99 ± 0·03
0·7	4·63	2·97	2·06	1·51 ± 0·06	0·91 ± 0·05
0·8	4·51	2·88	1·96	1·35 ± 0·15	0·75 ± 0·15
0·9	4·34	2·75	1·85	—	—
1·0	4·04	2·53	1·71	—	—

† The uncertainties in the estimates of kT_c/J for the cubic lattices are within ±0·01.

The critical temperatures kT_c/J for the plane triangular, simple quadratic and the cubic lattices are shown in table 1. The critical temperatures for other lattices are not shown here because the uncertainties in the estimates are very large. The uncertainty in the estimate of the critical parameters increases when the coordination number of the lattice q decreases, or when the anisotropy parameter η changes from the Ising limit to the isotropic Heisenberg limit. We see from table 1 that the critical temperature of the anisotropic Heisenberg model decreases as the coordination number decreases, or when the isotropy varies from the Ising to the isotropic limit. The present estimates for the critical temperatures are much lower than those obtained by the effective field theories and by the Green function technique. Dalton and Wood (1967) adopted the hypothesis that $\gamma = 1{\cdot}75$ and 1·25 in the two- and three-dimensional lattices respectively for $0 \leqslant \eta < 1$, and estimated T_c from the five-term series by using $(\chi)^{1/\gamma}$. It is interesting to note that their estimates are in good agreement with the present results obtained from the seven-term series without assuming any specific value of γ in the estimate. This provides evidence which supports, to some extent, the universality hypothesis.

Table 2. Critical indices γ for the FCC and the BCC lattices

Anisotropy η	FCC lattice	BCC lattice
0	1·25 ± 0·01	1·24 ± 0·02
0·1	1·25 ± 0·01	1·24 ± 0·02
0·2	1·25 ± 0·01	1·24 ± 0·03
0·3	1·24 ± 0·02	1·23 ± 0·03
0·4	1·24 ± 0·02	1·23 ± 0·03
0·5	1·24 ± 0·02	1·22 ± 0·04
0·6	1·23 ± 0·03	1·21 ± 0·04
0·7	1·23 ± 0·03	1·21 ± 0·04
0·8	1·23 ± 0·03	1·21 ± 0·04
0·9	1·24 ± 0·03	1·22 ± 0·04
1·0	1·38 ± 0·03	1·38 ± 0·04

For the critical indices the uncertainties in the estimates are much larger. In table 2 we show only the values for the FCC and the BCC lattices. Results for other lattices are too erratic to be included here. Our results are consistent with the hypothesis that γ remains constant ($\cong 1{\cdot}25$) for $0 \leqslant \eta < 1$ and changes discontinuously at $\eta = 1$.

5. Summary and discussion

The high temperature susceptibility series through order T^{-6} for the spin-S anisotropic Heisenberg model, recently obtained by Wood and Dalton (1972), has been extended to order T^{-7} for spin-$\frac{1}{2}$ systems. The critical temperature and the critical index are estimated from the series for a ferromagnet with anisotropy varies from the Ising limit to the isotropic Heisenberg limit. The critical temperature of the anisotropic Heisenberg model decreases as the coordination number decreases, or when the anisotropy parameter varies from 0 to 1. A hypothesis concerning the critical index states that the index remains constant for $0 \leqslant \eta < 1$, and changes discontinuously at $\eta = 1$ (Griffiths 1970, Fisher 1966). Although the discontinuity of the critical index cannot be shown exactly from a finite number of terms in the high temperature series, our estimates are consistent with the hypothesis. Since the results for $s = \infty$ are also consistent with the hypothesis, it is reasonable to state that the hypothesis is true for all spin values.

For the two-dimensional lattices the existence of nonzero critical temperatures for the isotropic Heisenberg model has been a subject of speculation (Stanley and Kaplan 1966). A related question which arises here is that, should the isotropic Heisenberg model not have a phase transition, what largest value of η would be permitted in order to have a phase transition? It is known that phase transition occurs for a two-dimensional Ising model (Yang 1952). From equation (7) it is clear that an anisotropic ferromagnet will have a nonzero critical temperature for $\eta \ll 1$. We have estimated the critical temperature from our series for the two-dimensional lattices. Although the estimates are less accurate, there is evidence that the critical temperatures are positive for η as large as 0·8 for the simple quadratic and the triangular lattices. We can not make any conclusion for larger values of η. Presumably, phase transition will occur in the spin-$\frac{1}{2}$ two-dimensional lattices for all values of η less than unity. It has been suggested by Stanley (1971) that a conformal transformation of the series to $K^* = K/(1 + tK)$ may reduce the uncertainties in the estimates of the critical parameters. We find that estimates obtained by the transformation method are also unreliable because the estimates of the critical parameters depend strongly on the parameter t. It is then necessary that more terms in the susceptibility be obtained. By the method we used in this paper, it seems possible that the present series can be extended to order T^{-9} if a fast computer with large capacity (able to handle matrix calculation of dimension 126×126) is available.

In this paper the estimate of the critical parameters is confined to $J > 0$ and $0 \leqslant \eta \leqslant 1$. To determine the critical parameters for other values of J and η, we have to consider the series expansions for χ_{xx} and for the staggered susceptibilities. These series are being studied.

References

Allan G T A and Betts D D 1967 *Proc. Phys. Soc.* **91** 341–52
Baker G A Jr 1961 *Phys. Rev.* **124** 768–74
—— 1963 *Phys. Rev.* **129** 99–102

Baker G A Jr, Gilbert H E, Eve J and Rushbrooke G S 1967 *Phys. Rev.* **164** 800–17
Betts D D, Elliott C J and Lee M H 1970 *Can. J. Phys.* **48** 1566–77
Chen H H and Joseph R I 1972 *J. math. Phys.* **13** 725–39
Chen H H and Levy P M 1973 *Phys. Rev.* B **7** 4284–89
Dalton N W and Wood D W 1967 *Proc. Phys. Soc.* **90** 459–74
Domb C 1960 *Adv. Phys.* **9** 149–361
Domb C and Sykes M F 1962 *Phys. Rev.* **128** 168–73
Domb C and Wood D W 1965 *Proc. Phys. Soc.* **86** 1–14
Fisher M F 1959 *Physica* **25** 521–4
—— 1966 *Phys. Rev. Lett.* **16** 11–14
Griffiths R B 1970 *Phys. Rev. Lett.* **24** 1479–82
Jasnow D and Wortis M 1968 *Phys. Rev.* **176** 739–50
Jou D C and Chen H H 1973 *Phys. Lett.* A **43** 79–80
Obokata T, Ono I and Oguchi T 1967 *J. Phys. Soc. Japan* **23** 516–21
Rushbrooke G S and Wood P J 1958 *J. molec. Phys.* **1** 257–83
Smart J S 1966 *Effective Field Theories of Magnetism* (London: Saunders) ch 3 and 4
Stanley H E 1971 *Phys. Rev.* B **4** 1613–30
Stanley H E and Kaplan T A 1966 *Phys. Rev. Lett.* **17** 913–15
Wood D W and Dalton N W 1972 *J. Phys. C: Solid St. Phys.* **5** 1675–86
Yang C N 1952 *Phys. Rev.* **85** 808–16