PARKER INSTABILITY IN A REALISTIC GRAVITATIONAL FIELD

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ABSTRACT

Magnetic fields with an ordered component parallel to the plane permeate the disks of spiral galaxies and may do the same in the gaseous disks of protostellar and mass-transfer binary star systems. Magnetic buoyancy makes such configurations prone to the Parker instability, which, combined with the magnetorotational instability recently rediscovered by Balbus & Hawley in this journal, may lead to sustained dynamo action and other interesting magnetohydrodynamic phenomena. As a prelude to more complete studies, we analyze the simpler problem of Parker's instability in a disk with a realistic vertical structure, but without including the effects of rotation and shear in the horizontal directions. In addition to the continuum modes found by previous authors for the idealized case when the vertical gravitational field is taken to be a simple step function, we discover the possibility for discrete modes whose power is more spatially confined in z. When these discrete modes prove unstable, they favor condensations that are placed antisymmetrically with respect to the midplane, a feature found also in numerical simulations carried into the nonlinear regime by Matsumoto et al. Transport motions across the midplane would help to alleviate one criticism directed against the convective model of accretion disk viscosity developed by Lin and his colleagues. For the continuum modes, we note that the characteristic length scale for instability is typically half of the conventionally estimated value, yielding growth rates that are approximately double previous estimates.

Subject headings: instabilities — ISM: magnetic fields — MHD

1. INTRODUCTION

1.1. Historical Review

The interstellar medium of the Galaxy consists of (dusty) neutral and ionized gas, magnetic fields, and cosmic rays, which all contribute to the total pressure. In some average sense, the pressure gradient perpendicular to the plane of the disk balances the vertical component of the gravitational force per unit volume. Confinement of relativistic particles to the Galactic plane can then be understood as resulting from a configuration in which the systematic component of the magnetic field lies primarily horizontally in the plane of the Galaxy, pointing more or less in the circular direction. If we adopt a cylindrical coordinate system with origin at the disk center and with \( z = 0 \) defining the equatorial plane, we assume the dominant component of \( B \) to be \( B_\phi \). Polarization measurements of the radio emission from external spiral galaxies verify this general expectation, provided that we do not make too big a distinction over the difference between the \( \phi \)-direction and the direction of spiral arms, and provided that we ignore the enhancement of the turbulent part of the field structure in spiral arms due to star formation and other violent activity (see, e.g., the review of Krause 1990).

In a series of seminal papers, however, Parker (1966, 1967a, b) demonstrated that this equilibrium is subject to a basic instability, the magnetohydrodynamic (MHD) analog of the classical Rayleigh-Taylor instability involving superposed fluids in a vertical gravitational field (cf. Chandrasekhar 1961). In Parker's problem, the two "light fluids"—cosmic rays and magnetic field—are held down by the weight of a "heavy fluid"—the thermal components of the interstellar gas—and the system tends to overturn, with the cosmic rays inflating the field into magnetic mountains, and with the thermal gas sliding down into the magnetic valleys (see, e.g., Mouschovias' 1974 calculations of the nonlinear resolution of the instability). The instability will arise whenever the sum of the magnetic and cosmic-ray pressures divided by the gas pressure exceeds \( \gamma - 1 \), where the effective adiabatic index \( \gamma \) characterizes the fractional increase of the gas pressure \( P \) with the fractional increase of its mean density \( \rho \) (see eq. [3.7] below).

Mouschovias, Shu, & Woodward (1974; see also Assen et al. 1978; Blitz & Shu 1980; Matsumoto et al. 1988, 1990) proposed that the triggering of Parker's instability in galactic spiral arms yields the formation of giant molecular clouds (GMCs); however, Cowie (1981) and Elmegreen (1982, 1992) have plausibly argued that self-gravity of the gas layer plays at least as important a role, especially for the largest cloud complexes. On the other hand, our new analysis in §3 suggests a numerical correction of a factor of 2 (or more) in the effective scale length and growth rates for Parker's instability. This numerical correction would lower the parameter \( s_0 \) defined by Elmegreen (1982) by a factor of 4 (or more). Thus the contribution of gas self-gravity to the Parker-Jeans instability may be less important than thought previously.

The effects of rotation on Parker's instability, in a geometry appropriate for galactic and accretion disks, were considered by Shu (1974) and by Zweibel & Kulsrud (1975). These authors found that uniform rotation could lower the growth rates of unstable modes, but it would not change the criterion for the onset of the instability. Moreover, in the absence of diffusive effects, the growth
rates become largest at infinite radial wavenumbers, where, independent of the amount of rotation or shear, Shu recovered Parker's original dispersion relation derived in the absence of differential rotation.

Ryu & Goodman (1992) analyzed a complementary problem—without magnetic fields but with differential rotation included in the so-called shearing-sheet approximation. In this situation, Parker's instability corresponds to ordinary thermal convection, a leading contender in the search for a mechanistic explanation for the anomalous viscosity of non-self-gravitating accretion disks (see, e.g., Lin & Papaloizou 1980). In their study of the problem, Ryu & Goodman find the surprising result that convective instability in the presence of differential rotation transports angular momentum (in the small-amplitude regime) down the equilibrium gradient not of the angular velocity but of the specific angular momentum, i.e., in the opposite sense to that required to make viscous accretion disks work.

In the interim, Balbus & Hawley (1991) have resurrected, for application to accretion disks, a magnetorotational instability discovered by Chandrasekhar (1960). The original calculations were performed in a geometry that corresponds to a differentially rotating infinite cylinder of electrically conducting gas threaded in the equatorial plane by a vertical magnetic field (although the growth of axisymmetric perturbations is insensitive to an arbitrary level of toroidal field $B_\phi$). When the square of the gas angular velocity, $\Omega^2$, decreases with distance $r$ from the rotation axis, torsional Alfvén waves propagating in the $z$-direction can become unstable because alternating annuli of gas in the vertical direction tend to torque down (or torque up) their neighbors in such a manner as to enhance the original sense of displacement. The instability arises whenever the magnetic pressure $B_z^2/8\pi$ associated with the vertical component of the magnetic field is relatively small compared with the thermal gas pressure $P$; otherwise the resulting tension of bent field lines in the meridional plane will tend to restore the inwardly contracting (or outwardly expanding) annuli to their original positions.

In numerical simulations of the instability carried into the nonlinear regime, Hawley & Balbus (1991) found that the action of the magnetorotational instability on a weak magnetic field directed initially in the $z$-direction produces stretching that amplifies the horizontal components $B_\theta$ and $B_z$ (if initially weak), with an attendant vigorous transport of angular momentum down the radial gradient of angular velocity that bodes most promisingly for application to accretion disks. In more recent studies of the (nonaxisymmetric) initial-value problem in a shearing-cylinder geometry, Balbus & Hawley (1992) and Hawley & Balbus (1992) also find impressive (but nonexponential) growth associated with configurations that start with purely radial fields $B_\phi$. Unfortunately (as the authors themselves note), the significance of this result in terms of stability has a less clear interpretation than the earlier normal-mode analysis, since (a) the initial state does not correspond to an equilibrium state and (b) the growth is transient. (It is always possible to superpose a complete set of purely oscillatory normal modes to give any finite amount of transient growth to a particular disturbance—e.g., a shearing wavelet.)

Balbus & Hawley state as part of their plans for future work the inclusion of stratification in the $z$-direction to study the role of magnetic buoyancy in limiting the growing strength of the magnetic fields amplified by shearing motions. Here we wish to point out another role for magnetic buoyancy: the regeneration of $B_z$ via Parker's instability to feed back on the magnetorotational instability part of the picture, the entire process—with the inclusion of turbulent dissipation and reconnection—contributing perhaps to the self-sustaining disk dynamo that Balbus & Hawley rightly consider the most exciting possibility for the action of the magnetorotational instability (see also Tout & Pringle 1992).

If the concepts of MHD do apply, the two instabilities may combine very naturally to drive accretion in a magnetized differentially rotating disk. For example, to avoid the instability discussed by Balbus & Hawley, one might try to arrange $B$ in the initial (equilibrium) state to lie wholly in the $\phi$-direction (rationalizing this choice on the basis of the stretching otherwise produced by differential rotation). To avoid Parker's instability, the ratio $z$ of magnetic pressure $B_z^2/8\pi$ to gas pressure $P$ must be less than a certain critical value. But such a quiescent disk would have no internal energy sources (due, e.g., to accretion). Therefore, it would radiatively cool, dropping the gas pressure $P \propto \rho T$ at any given density $\rho$, and thereby raising the ratio $x (B \propto \rho$ in one-dimensional compression if we assume field freezing, so $z \propto B^2/\rho T$ increases even if the gas compresses vertically on cooling), until $x$ rises above the critical value needed for the onset of Parker's instability (i.e., until the disk becomes magnetically convective by cooling from its surface: cf. Lin & Papaloizou 1980). The onset of Parker's instability then yields the weak poloidal fields, $B_\phi$ and $B_z$, that drive the magnetorotational instabilities discussed by Balbus & Hawley. The resulting disk accretion can raise the gas temperature and pressure until the system settles into a (turbulent) state in which $x$ is thermostated to a level where Parker's instability barely regenerates ordered poloidal fields at a rate needed to feed into the magnetorotational part of the cycle to maintain quasi-steady macroscopic conditions (in the face of reconnection and dissipation).

Unfortunately, two flaws mar this otherwise attractive scenario—one puzzling, the other possibly fatal. The puzzle concerns how instabilities of an intrinsically intermediate scale (vertical scale height) ever manage to generate magnetic fields possessing global order. In particular, how does the local shear predominantly stretch poloidal fields of the right orientation to reinforce the strength of the mean (global) toroidal field? Perhaps, as many people have speculated in recent years, density-wave streaming motions provide the background to produce the global field patterns that astronomers observe in many spiral galaxies (see, e.g., Krause 1990 and references therein). In contrast, no empirical evidence exists in non-self-gravitating accretion disks that they need to possess globally ordered fields.

The possibly fatal flaw concerns the ionization balance in realistic astrophysical disks. Detailed calculations in minimum-mass models of the protosolar nebula cast deep doubts on the assumption that the disks surrounding young sunlike stars could be sufficiently electrically conducting, except in their surface layers, as to justify, by several orders of magnitude, the assumptions of ideal magnetohydrodynamics (Umebayashi & Nakano 1988; Nakano 1992). Within the large uncertainties involved in the complicated issue of recombination in the presence of charged grains, these authors find that nebular disks will not couple dynamically to magnetic fields except in their innermost regions (say, within 5 stellar radii if we restrict our attention to disks with dimensions $\leq 100$ AU), where the thermal ionization of metals can provide the requisite ionization levels. Other astrophysical situations may possess more favorable ionization levels, and we ignore this potential difficulty in the reexamination of Parker's instability mechanism that follows.
1.2. Motivation for Present Study

To simplify the analysis, most previous studies have followed Parker (1966) in assuming a step function for the vertical gravitational field, rising discontinuously from one uniform value, \( g_z = g_0 \), below the midplane to its opposite uniform value, \( g_z = -g_0 \), above \( z = 0 \). In this work we adopt a more realistic form of gravitational field, consistent, in the galactic context, with that supplied by an isothermal slab of stars, \( g_z = -g_0 \tanh(z/2H_\star) \) (see eq. [2.4b]). For accretion disks dominated by the gravitational field of the massive central object, a Taylor series expansion for small \( z \) would yield a linear dependence on \( z \), \( g_z = -\Omega^2 z \). The numerical simulations of Matsumoto et al. (1988, 1990) focused essentially on this example. In the current paper, we shall choose for definiteness to discuss the case \( g_z = -g_0 \tanh(z/2H_\star) \). We can recover the case \( g_z \propto -z \) by taking the limit \( H_\star \to 0 \) and \( g_0 \to \infty \) with \( g_0/2H_\star \) finite (i.e., the limit \( R \to \infty \) in \S 2.2).

A clear motivation exists for abandoning Parker’s simplified step function: to avoid the singularity across \( z = 0 \), which would prevent us from examining the interesting question of model parity. An odd mode has vertical fluid velocity \( u_z = 0 \) at \( z = 0 \). There are then no midplane crossings, which leads—at lowest order—to density condensations (magnetic valleys) located exactly on the midplane (implicitly assumed by Parker 1966 and Mouschovias 1974). An even mode would contain midplane crossings and condensations that alternate above and below \( z = 0 \) (see, e.g., Figs. 2, 4, 6, 8 of Matsumoto et al. 1990). Magnetic Rayleigh-Taylor instabilities of relatively large scale may have a preference for this “bending” mode of behavior (especially in the case, \( g_z \propto -z \), see \S 3), because staggering the locations of condensations yields slightly more room for the instability to proceed. It is interesting to note that the actual distribution of the GMCs in the Galaxy exhibit both kinds of parities (Blitz 1980).

In a different context, Ruden, Papaloizou, & Lin (1988) found that, in the absence of magnetic effects, convective instabilities of realistic models of primitive solar nebulae generate weak or no motions across \( z = 0 \), because buoyant forces vanish at the midplane. This observation may make it difficult for accretion driven by convective turbulence (even if it avoids the paradox posed by Ryu \\& Goodman’s (1992) linearized calculation) to tap into the main energy reservoir of the disk. A further problem arises because the dominant modes of convective instability in a strongly differentially rotating disk tend to produce long skinny columns of upwelling and downwelling gas (Shu 1974), cutting into the efficiency of the process for angular momentum transport (W. Cabot \\& P. Cassen 1991, private communication). The inclusion of magnetic forces may help to alleviate some of these difficulties.

Apart from the question of modal parity, our reanalysis of Parker’s instability shows two types of modes. The first set has the same dispersion relation as when \( g_z \) equals a step function. In this case, a continuum of wavenumbers in the vertical direction is possible for a given wavenumber in the horizontal direction. These continuum modes do not discriminate between odd and even parity.

The second set of modes has only a finite number of oscillations in the \( z \)-direction for a given horizontal wavenumber. We call these disturbances discrete modes. When present, the lowest member of this set is favored because it proves the most difficult to stabilize and leads to the highest growth rates. This mode has even parity (i.e., it contains midplane crossings). The existence of the discrete modes is limited to regions of parameter space where the magnetic field is fairly weak. In circumstances more favorable to the rapid growth of the Parker instability, we find that the discrete modes merge into the continuous spectrum.

2. FORMULATION OF THE PROBLEM

2.1. Equilibrium State

To obtain an analytically tractable, we follow Parker in making several simplifying assumptions. First, we ignore the effects of differential rotation altogether. This allows us to use local Cartesian coordinates \((x, y, z)\) instead of cylindrical coordinates \((\sigma, \varphi, z)\). Consistent with this approximation, we assume an equilibrium state that is stratified only in \( z \) and has magnetic fields pointing only in \( y \) (the local tangential direction): \( B_y = B_y(x) \). We further assume that the equilibrium gas pressure \( P_0 \) at every height \( z \) scales directly with the equilibrium gas density \( \rho_0 \):

\[
P_0 = a^2 \rho_0 \, ,
\]

where \( a \) is the effective isothermal sound speed. Moreover, we postulate that the equilibrium magnetic and cosmic-ray pressures form constant ratios, \( \alpha \) and \( \beta \), with respect to the gas pressure at every \( z \):

\[
\frac{B_z^2}{8\pi} = \alpha P_0 \, ,
\]

\[
P_{\text{ero}} = \beta P_0
\]

We do not follow Parker in his simplified representation of the Galactic gravitational field. Instead, we choose the form appropriate for an isothermal slab of stars (cf. Spitzer 1942; Camm 1949):

\[
\rho_\star = \frac{g_\star}{4H_\star} \text{sech}^2 \left( \frac{z}{2H_\star} \right),
\]

\[
g_\star = -g_0 \tanh \left( \frac{z}{2H_\star} \right),
\]

where

\[
g_0 = 2\pi G \rho_\star \, ,
\]

\[
H_\star \equiv \left\langle \frac{c_s^2}{g_0} \right\rangle
\]
and $\sigma_z$ and $\langle c_z^2 \rangle^{1/2}$ are respectively, the local surface density of the disk stars and their rms random velocity in the $z$-direction. Notice that only in the asymptotic limits, $z \to \pm \infty$, where the volume density of the stars drops off exponentially with height, $\rho_\ast \propto e^{-|z|/H_\ast}$, do we recover the step-function values $g_z = \mp g_0$ assumed by Parker in his original analysis.

For simplicity, we ignore the self-gravity of the interstellar medium. The condition for hydrostatic equilibrium of the mixture of thermal gas, magnetic field, and cosmic rays then becomes

$$a^2(1 + \alpha + \beta) \frac{d\rho_0}{dz} = -\rho_0 g_0 \tanh \left( \frac{z}{2H_\ast} \right). \quad (2.6)$$

Equation (2.6) may be integrated to give

$$\rho_0(z) = \rho_0(0) \operatorname{sech}^{2R} \left( \frac{z}{2H_\ast} \right) = \rho_0(0) \operatorname{sech}^{2R} \left( \frac{z}{2RH_\ast} \right), \quad (2.7)$$

$$R \equiv \frac{\langle c_z^2 \rangle}{(1 + \alpha + \beta)a^2} = \frac{H_\ast}{H}, \quad (2.8a)$$

$$H \equiv (1 + \alpha + \beta)a^2 g_0. \quad (2.8b)$$

Notice that $R$ measures the ratio of the square of the effective velocity dispersion in the vertical direction for the disk stars to the analogous quantity in the combined interstellar medium. The (exponential) length scale $H$ can be identified as the asymptotic scale height of the interstellar gas, since $\rho_0(z) \propto e^{-|z|/H}$ as $|z| \to \infty$. If $R \gg 1$, $\rho_0(z)$ behaves like a Gaussian, $\rho_0(z) \approx \rho_0(0) \exp \left(-z^2/4RH^2\right)$, for $|z| \ll 2R^{1/2}H$.

We obtain the equivalent half-thickness of the interstellar gas through the integration

$$H_{\text{eq}} \equiv \frac{1}{\rho_0(0)} \int_0^\infty \rho_0(z) dz = 2^{2R^{-1}} R \frac{\Gamma(2R)}{\Gamma(2)} H, \quad (2.9)$$

where $\Gamma$ is the gamma function. The equivalent half-thickness of the stellar disk is given by a similar formula with $R$ taken to be unity, i.e., $H_{\ast\text{eq}} = 2H_\ast$. Figure 1 gives the ratio $R_{\text{eq}} = H_{\ast\text{eq}}/H_{\text{eq}}$ as a function of the ratio $R = H_\ast/H$. In the solar neighborhood $R_{\text{eq}} \approx 2$, corresponding to $R \approx 3.5$.

### 2.2. Small-Amplitude Perturbations

The ideal MHD equations governing the time-dependent behavior of partially ionized gas, magnetic field, and cosmic rays in a gravitational field have their usual forms. In particular, we suppose that the gas responds to dynamic compressions in such a way that the Lagrangian change of its pressure $P$ varies as the $7$th power of its Lagrangian change of density $\rho$. To simulate the effects of cooling, or imperfectly understood effects such as interstellar turbulence and tangled magnetic fields, we allow $\gamma$ to be both less than and greater than unity (see the discussion of Zweibel & Kulsrud 1975 on the complications in arriving at an educated guess for $\gamma$). Following Parker, we approximate the cosmic-ray pressure to have no gradients along field lines: $B \cdot \nabla P_{cr} = 0$. In the linearized limit, this restriction is equivalent to the statement that the substantial derivative of $P_{cr}$ equals zero (Shu 1974); i.e., on the time scale

![Fig. 1.—Ratio $R_{\text{eq}}$ of equivalent thicknesses for stellar and gas disks vs. ratio $R$ of asymptotic exponential thicknesses](image-url)
for the development of Parker's instability, we assume that the cosmic rays have no sources or sinks of pressure as we follow the motion of the thermal gas.

If we use the subscript 1 to denote disturbance quantities, we can introduce dimensionless perturbation variables by defining $s = \rho_1/\rho_0, \mathbf{u} = \mathbf{u}_1/\mathbf{u}_0, \mathbf{b} = \mathbf{B}_1/\mathbf{B}_0, p = P_1/P_0,$ and $p_{ce} = P_{ce1}/P_{ce0}$. We also define the dimensionless spacetime coordinates $(x', y', z', t') = (x/H, y/H, z/H, at/H)$. Keeping only first-order terms and dropping the primes for notational simplicity, we obtain, for the linearized perturbation equations,

\[
\begin{align*}
\frac{\partial s}{\partial t} - \Theta u_x + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} &= 0, \\
\frac{\partial u_x}{\partial t} + \frac{\partial p}{\partial x} + \beta \frac{\partial p_{ce}}{\partial x} - 2\alpha \left(\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z}\right) &= 0, \\
\frac{\partial u_x}{\partial t} + \frac{\partial p}{\partial x} + \beta \frac{\partial p_{ce}}{\partial x} + \Theta \beta b_x &= 0, \\
\frac{\partial u_x}{\partial t} + \Theta(1 + \alpha + \beta s - \Theta(p + \beta p_{ce}) + \frac{\partial p}{\partial z} + \beta \frac{\partial p_{ce}}{\partial z} - 2\alpha \left(\frac{\partial b_z}{\partial y} + \Theta b_y - \frac{\partial b_y}{\partial z}\right) &= 0, \\
\frac{\partial b_z}{\partial t} - \frac{\partial u_z}{\partial y} &= 0, \\
\frac{\partial b_x}{\partial t} + \frac{\partial u_x}{\partial x} - \frac{\Theta}{2} u_z + \frac{\partial u_z}{\partial z} &= 0, \\
\frac{\partial b_z}{\partial t} - \frac{\partial u_z}{\partial x} &= 0, \\
\frac{\partial p}{\partial t} - \Theta u_z + \gamma \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) &= 0, \\
\frac{\partial p_{ce}}{\partial t} - \Theta u_x &= 0.
\end{align*}
\]

In the above equations, $\Theta$ denotes the quantity

$$\Theta = \tanh(z/2R),$$

and equals the ratio of the gravitational field at a given point to its maximum value at infinity. If $\Theta$ is set equal to unity 1 (or, equivalently, if we let $R \to 0$), equations (2.10a)–(2.10i) become equivalent to those investigated by Parker (1967b). The case where $q_z$ varies linearly with $z$ corresponds to letting $R$ become very large.

Since the coefficients of equations (2.10a)–(2.10i) do not depend on $x, y,$ or $t$, we may look for eigensolutions of the form

$$\begin{bmatrix}
  s(x, y, z, t) \\
  u(x, y, z, t) \\
  b(x, y, z, t) \\
  p(x, y, z, t) \\
  p_{ce}(x, y, z, t)
\end{bmatrix} = e^{\sigma + i(\xi x + \eta y)},$$

with the real part of $\sigma$ yielding the nondimensional growth rate (in units of $a/H$), and with $\xi$ and $\eta$ being wavenumbers (in units of $1/H$) in the $x$- and $y$-directions. We specify $\xi$ and $\eta$ to be real but allow $n$ to be complex. No confusion should arise from our using the same symbols (e.g., $p_{ce}$) for a perturbation quantity and its $z$-dependent part.

With the substitution of equation (2.12), equations (2.10a)–(2.10i) become algebraic or ordinary differential equations in $z$:

\[
\begin{align*}
\frac{du_x}{dz} - \Theta u_z + ns + i\xi u_x + inu_y &= 0, \\
u_x + i\xi(p + \beta p_{ce}) - 2\alpha i(nb_z - \xi b_x) &= 0, \\
u_y + in(p + \beta p_{ce}) + \Theta \beta b_x &= 0, \\
\frac{d}{dz} (p + \beta p_{ce} + 2\alpha b_x) + nu_z + \Theta(1 + \alpha + \beta s - \Theta(p + \beta p_{ce}) - 2\alpha(\Theta b_y + inb_y) &= 0, \\
\frac{nb_z - inu_x}{nz} &= 0.
\end{align*}
\]
\[ \frac{du_z}{dz} + nb_z + i\xi u_z - \frac{\Theta}{2} u_z = 0 , \]  
\[ nb_z - i\eta u_z = 0 , \]  
\[ \gamma \frac{du_z}{dz} + np - \Theta u_z + i\gamma (\xi u_z + \eta u_z) = 0 , \]  
\[ np_{er} - \Theta u_z = 0 . \]  

Notice that the only derivatives which enter are those of the z-component of the gas velocity \( u_z \) and of the total perturbational pressure \( p_{tot} \equiv p + \beta p_{er} + 2\alpha b_z \). When we solve equations (2.13a)–(2.13i) for these terms, we obtain the matrix equation
\[ \frac{d}{dz} \begin{pmatrix} u_z \\ p_{tot} \end{pmatrix} + \begin{pmatrix} -A\Theta & B \\ C + D\Theta^2 & (\text{A} - 1)\Theta \end{pmatrix} \begin{pmatrix} u_z \\ p_{tot} \end{pmatrix} = 0 , \]  
where
\[ A \equiv \frac{(1 + \alpha + \beta)n^2}{\Delta} , \]  
\[ B \equiv n \left( \frac{n^2 + \gamma n^2}{\Delta} + \frac{\xi^2}{n^2 + 2\alpha n^2} \right) , \]  
\[ C \equiv \frac{n^2 + 2\alpha n^2}{n} , \]  
\[ D \equiv \frac{(1 + \alpha + \beta)}{n} \left[ 1 - \frac{(1 + \alpha + \beta)(n^2 + 2\alpha n^2)}{\Delta} \right] , \]  
with \( \Delta \) defined by
\[ \Delta \equiv (2\alpha + \gamma)n^2 + 2\alpha\gamma n^2 . \]  

We may eliminate \( p_{tot} \) from the two coupled first-order equations (2.14) to obtain a single second-order equation for \( u_z \). The transformation
\[ u_z \equiv \Psi \cosh^8 (\xi/2R) \]  
then puts the resulting ordinary differential equation into normal form:
\[ \frac{d^2\Psi}{dz^2} + \left[ E + V_0 \operatorname{sech}^2 \left( \frac{\xi}{2R} \right) \right] \Psi = 0 , \]  
where the constants \( E \) and \( V_0 \) depend on the physical parameters \( \alpha, \beta, \gamma, R \) and the squares of the growth rate and horizontal wavenumbers, \( n^2, \xi^2, \eta^2 \), as follows:
\[ -E = \frac{1}{\Delta} \left[ \frac{2\alpha(1 + \alpha + \beta)\xi^2 n^2}{n^2 + 2\alpha n^2} + n^4 + (2\alpha + \gamma)(\xi^2 + \eta^2) n^2 + 2\alpha\gamma(\eta^2 - \eta^2) (\xi^2 + \eta^2) \right] + \frac{1}{4} , \]  
\[ V_0 = \frac{1}{\Delta} \left[ \frac{2\alpha(1 + \alpha + \beta)\xi^2 n^2}{n^2 + 2\alpha n^2} - 2\alpha\gamma\eta^2 (\xi^2 + \eta^2) - \frac{(1 + \alpha + \beta)n^2}{2R} \right] + \frac{1}{4} + \frac{1}{4R} , \]  
In equations (2.19a) and (2.19b) we have introduced the square of the Parker wavenumber,
\[ \eta^2 \equiv \frac{(1 + \alpha + \beta)(1 + \alpha + \beta - \gamma)}{2\gamma} , \]  
and \( \Delta \) is defined by equation (2.16).

2.3. Analogy with Quantum Oscillator

Except for constants, equation (2.18) is identical to Schrödinger's equation for a particle with energy \( E \) confined to a \( \operatorname{sech}^2 \) potential, with a height of \( V_0 \) and a half-width \( 2R \). The corresponding problem in quantum mechanics has been solved by P. M. Morse (see the exposition by Landau & Lifshitz 1958, pp. 69–70; see also Vandervoort 1970 for a different application). Two sets of solutions exist. For \( E > 0 \) the solutions behave similarly to waves scattering off a potential barrier (if \( V_0 < 0 \)) or well (if \( V_0 > 0 \)). This scattering problem has a continuum of eigensolutions; i.e., for any given set of horizontal wavenumbers, \( \xi \) and \( \eta \), there exists an
oscillatory disturbance with a vertical wavenumber at infinity that approaches any real value \( \zeta \). We call these solutions the continuum modes.

For \( E < 0 \) the solutions look like the bound states of an anharmonic oscillator. Since the potential well has finite depth and breadth, there can exist only a finite number of these bound states; i.e., for a given set of horizontal wavenumbers \( \xi \) and \( \eta \), only a finite number of modes with oscillatory behavior confined in the \( z \)-direction are possible. We call these solutions the discrete modes. In the next section we explore the behavior of the two kinds of modes.

3. ANALYTICAL RESULTS

3.1. Continuum Modes

The properties of the continuum modes, when \( E > 0 \), may be summarized by their behavior at infinity. In the limit \( |z| \to \infty \), equation (2.18) becomes

\[
\frac{d^2 \Psi}{dz^2} + E \Psi = 0, \tag{3.1}
\]

whose normalized solutions read

\[
\Psi = e^{i \zeta z}, \quad \text{with} \quad \zeta = \pm E^{1/2}. \tag{3.2}
\]

If we regard \( \zeta^2 \) as freely specifiable, the relation \( E = \zeta^2 \), with \( E \) given by equation (2.19a), corresponds to a cubic equation for \( n^2 \):

\[
n^6 + N_2 n^4 + N_1 n^2 + N_0 = 0, \tag{3.3}
\]

where

\[
N_2 = 2\alpha \eta^2 + (2\alpha + \gamma)(k^2 - \frac{4}{3}), \tag{3.4a}
\]

\[
N_1 = 4\alpha(\alpha + \gamma)(k^3 - \frac{1}{3})\eta + 2\alpha[(1 + \alpha + \beta)\xi^2 - \gamma\eta^2(\xi^2 + \eta^2)], \tag{3.4b}
\]

\[
N_0 = 4\alpha^2\gamma(16k^3 - \frac{1}{3})\eta^2 - \eta^2(\xi^2 + \eta^2). \tag{3.4c}
\]

In the above, \( k^2 \) denotes the square of the total wavenumber at infinity,

\[
k^2 = \xi^2 + \eta^2 + \zeta^2, \tag{3.5}
\]

and the \( \frac{1}{3} \) in \( k^2 + \frac{4}{3} \) represents the usual effect of density stratification in an isothermal atmosphere (see, e.g., chap. 10 of Lamb 1916). In the limit of large wavenumbers, \( \xi, \eta, \) and \( \zeta \), the three roots of the dispersion relation for \( n^2 \) yield the usual modes of ideal MHD: the fast, slow, and Alfvén waves.

Equation (3.3) contains the dispersion relations derived by Parker (1966, 1967a, b) and Shu (1974) for the problem when \( g_z \) is a step function and differential rotation of the disk is ignored. The fast and Alfvén waves turn out to be stable at all wavenumbers. Instability (a positive root for \( n^2 \)) of the slow MHD wave arises when the combination of wavenumbers makes \( N_0 < 0 \), i.e., when

\[
\left( \frac{\eta^2}{\xi^2 + \eta^2} + \frac{1}{4} \right) < \eta^2 < \eta^2. \tag{3.6}
\]

For fixed \( \zeta^2 \) and \( \xi^2 \), the growth rate of unstable disturbances implied by equation (3.3) equals zero when \( \eta^2 = 0 \) and when \( \eta^2 = \) some upper limit (\( \eta^2 \) for large \( \xi^2 \)), reaching a maximum value for some intermediate (azimuthal) wavenumber scale (see Fig. 2 for some illustrative numerical examples). For the intermediate range to exist, we require \( \eta^2_0 > 0 \); this represents Parker's (1967b) criterion: instability arises if

\[
\alpha + \beta > \gamma - 1. \tag{3.7}
\]

If the effects of magnetic fields and cosmic rays can be ignored \( (\alpha = \beta = 0) \), equation (3.7) becomes equivalent to Schwarzschild's criterion: the medium is unstable to convection if the magnitude of the actual temperature gradient exceeds the magnitude of the adiabatic temperature gradient—in the present circumstances, if \( \gamma < 1 \).

For the continuum modes, then, the present analysis introduces only one qualitative modification and one quantitative modification of any importance. The qualitative one concerns the behavior of the solutions (in the form of hypergeometric functions) near \( z = 0 \). Symmetry allows the wave function \( \Psi \) to separate into solutions even and odd in \( z \). The continuation of such solutions to large \( |z| \) requires the complex superposition of the asymptotic forms \( e^{+i \xi z} \) and \( e^{-i \xi z} \), i.e., the superposition of propagating waves (when \( n \) is imaginary) to give standing waves of even and odd parity.

The concept of individual modes, however, loses considerable force when there exists a continuum of them. The more relevant problem in such circumstances is usually an initial-value problem, not a modal analysis, although the modal analysis retains its validity with regard to conclusions about exponential growth, since we have a complete set of functions. The primacy of an initial-value treatment holds especially forcefully in the present circumstances, where the constancy of \( \rho_0 |u_z|^2 \) at large \( |z| \) (cf. eqs.
[2.7] and [2.17]) implies that realistic continuum disturbances will acquire such large velocity amplitudes as probably to steepen into dissipative shocks at high altitudes (allowing “final” states lower in energy than “initial states”; see, e.g., Mouschovias 1974 and Matsumoto et al. 1990). In any case, our discussion demonstrates that the continuum modes exhibit no intrinsic preference for odd parity over even parity in terms of growth rates; the long-term manifestation of one over the other will depend on initial conditions or nonlinear effects.

The quantitative modification introduced by this paper involves the proper physical interpretation for the constant $g_0$, used to define length and time scales, $H$ and $H/\alpha$ (cf. eq. [2.8b]). In Parker’s (1966) idealization of $y$, above, or below the midplane as a constant $g_0$, he (and everyone who followed him) regarded it as natural to interpret $g_0$, in application to the Galaxy, as the mass-weighted average value $\langle g \rangle$ of $|g_z|$ over the gas layer. This procedure results in a value for $H \sim 150$ pc, and a wavelength of maximum growth rate $\sim 2\pi H \sim 1000$ pc. This paper demonstrates the correct choice for $g_0$ to be the asymptotic value of $|g_z|$ reached well above the disk. Since $g_0$ exceeds $\langle g \rangle$ typically by a factor of 2 in the solar neighborhood, the wavelength of maximum growth rate $\sim 2\pi H$ changes to $\sim 500$ pc, in better agreement with the mean spacing of GMCs in the solar neighborhood (Blitz 1980; Blitz & Shu 1980). In a similar fashion, all previous numerical estimates of growth rates need to be doubled; the resultant quadrupling of $n^2$ makes it that much harder for the self-gravity of the gas, which contributes to $n^2$ as $4\pi G\rho_0$, to compete in producing structures of an intermediate (Parker) scale (cf. Elmegreen 1982).

3.2. Discrete Modes

The discrete modes arise when $E < 0$. The eigenfunctions read

$$\Psi = \text{sech}' (z/2R) \mathcal{F} \left[ \epsilon - v, \epsilon + v + 1, \epsilon + 1, \frac{1}{2}[1 + \tanh (z/2R)] \right],$$  \hspace{1cm} (3.8)

where $\mathcal{F}$ is Gauss’s hypergeometric function, and the parameters $\epsilon$ and $v$ are defined by

$$\epsilon \equiv 2R(-E)^{1/2},$$  \hspace{1cm} (3.9a)

$$v \equiv 2R[(V_0 + \kappa^2)^{1/2} - \kappa],$$  \hspace{1cm} (3.9b)

with

$$\kappa \equiv \frac{1}{4R}.\hspace{1cm} (3.10)$$

The eigenfunction $\Psi$ will remain bounded as $|z|$ goes to infinity if the series expansion of the hypergeometric function truncates as a finite polynomial. This happens if

$$v - \epsilon = \ell, \hspace{1cm} (3.11)$$
where $\ell$ is a nonnegative integer, the order of the polynomial. Associated with $\ell$ we find it convenient to define two kinds of effective vertical wavenumbers, $\mu$ and $\nu$, such that

$$
\mu^2 = \frac{\ell(\ell + 1)}{4R^2}, \quad \nu^2 = \frac{(\ell + \frac{1}{2})^2}{4R^2}.
$$

The definitions (3.12), which allow only discrete sets of values, should be contrasted with that of equation (3.2), which allows a continuum of values.

For given integer $\ell$, equation (3.11) now results in a dispersion relation for the discrete modes because the quantities $\nu$ and $\epsilon$ are functions of the growth rate $n$, the horizontal wavenumbers $\xi, \eta$, and the physical parameters $\alpha, \beta, \gamma, R$. If we eliminate square roots in equations (3.9a), (3.9b), and (3.11), we obtain a fifth-order polynomial equation in $n^2$,

$$
n^{10} + N_4 n^8 + N_3 n^6 + N_2 n^4 + N_1 n^2 + N_0 = 0,
$$

where

$$
N_4 = 2p\xi^2 + 2(p + \alpha)\eta^2 + 2t\kappa,
$$

$$
N_3 = p^2 K^4 + 2[2(p + \gamma)\eta^2 + t\kappa](\xi^2 + \eta^2) + 4\alpha\kappa\eta^2 + 2t\mu^2\kappa + t^2\kappa^2 - p^2\nu^2,
$$

$$
N_2 = 2(p + \alpha)K^4\eta^2 + 4\eta(p + \gamma - \gamma)k^2\kappa\eta^2 + 4\eta^2[2q\eta^2 - 2pv^2 + (t - 2\gamma)(pv^2 + t\kappa^2)]\xi^2 + \eta^2,
$$

$$
N_1 = 2aq[(2p + \gamma)K^4 + 2(t - p - \gamma)c^2 + (2t - \gamma)\mu^2]n^4 - 8\alpha(2p\gamma - s\gamma - ps)v^2(\xi^2 + \eta^2)\eta^2,
$$

$$
N_0 = 2aq^2\eta^4[(K^4 - 2kk^2 - \mu^2)\eta^2 + 4\eta^4v^2(\xi^2 + \eta^2)]
$$

with

$$
k^2 \equiv \xi^2 + \eta^2 + \mu^2,
$$

$$
K^4 \equiv (\xi^2 + \eta^2)^2 + \mu^4,
$$

and

$$
p \equiv 2\alpha + \gamma,
$$

$$
q \equiv 2\alpha\gamma,
$$

$$
s \equiv 1 + \alpha + \gamma,
$$

$$
t \equiv 2 + 2\beta - \gamma.
$$

Because we have squared both sides of equations containing positive square roots, we can introduce spurious roots. Roots obtained from equation (3.13) therefore should be checked to make sure that they satisfy equation (3.11).

From equation (3.11) or (3.13), we find that the thicker the stellar disk compared with the gas (the larger the value of $R$), the more discrete modes the system can fit (the larger the run of $\ell$). The lowest order discrete mode, $\ell = 0$, is the only one that can exist for $R$ small. We call this model the fundamental discrete mode, and we note that it has even parity, i.e., fluid can cross the midplane. When other discrete modes also exist, the fundamental still grows the fastest.

In the polynomial of equation (3.13), the terms containing the highest powers of the horizontal wavenumbers, $\xi$ and $\eta$, all have positive coefficients. As a consequence, no unstable discrete mode exists at large $\xi$ or $\eta$. As we shall see below, when we increase $\xi$ or $\eta$, such discrete modes disappear, not by becoming stable (with $n^2 < 0$) but by merging into the continuum.

4. RELATION BETWEEN DISCRETE AND CONTINUOUS SPECTRUM

4.1. Merging into the Continuum

The discrete modes can exist only when $E$ is negative. To fix ideas, consider holding $\xi$ and $\ell$ fixed. Then, as we increase the (azimuthal) wavenumber $\eta$ along field lines, $E$ also increases algebraically and eventually reaches zero. At that point the growth rate $n$ (or frequency $-i\omega$ if the mode is stable) for the discrete mode has the same (generally nonzero) value as the growth rate (or frequency) for the continuum mode with the same horizontal wavenumbers $\xi$ and $\eta$ and a zero vertical wavenumber $\zeta$ ($\zeta = 0$ because $E = 0$; see eq. [3.2]). Further increases of $\eta$ result in the discrete mode (stable or unstable) disappearing into the continuum (stable or unstable).

The requirement that $E < 0$ corresponds to $v = \ell + \epsilon > \ell$, i.e., to

$$
V_0 > \frac{\ell(\ell + 1)}{4R^2}.
$$

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which is easiest to satisfy if $\ell = 0$. In other words, the fundamental will generally be the last to merge into the continuum if we imagine a sequence of events that slowly changes the disk conditions to those unfavorable for the existence of discrete modes (highest $\ell$ disappearing first). From definition (2.19b) for $V_0$ we can see that when $\eta^2$ is large, $V_0$ becomes less likely to satisfy the condition $V_0 > 0$. But large values of $\eta^2$ correspond to conditions that promote high growth rates of the continuum modes. Thus, discrete modes are not favored under conditions that lead to the fastest growth of Parker's instability.

Figures 3 and 4 illustrate these conclusions with some specific numerical examples: $(\alpha, \beta, \gamma) = (0.3, 0.0, 1.2)$ and $(0.2, 0.2, 1.3)$, respectively. Both cases have $R = 3.5$ and $\zeta^2 = 0$. In Figure 3, only the $\ell = 0$ discrete mode can be unstable; all other modes, including the continuum ones, are Parker-stable. In Figure 4, the $\ell = 0$ and $\ell = 1$ discrete modes are both unstable for an intermediate range of $\eta^2$, but the $\ell = 1$ mode is about to merge into the continuum of modes that have $\zeta^2 = 0$. Notice finally from the scales on the abscissa and the ordinate that the characteristic horizontal dimensions of instability in Figures 3 and 4 are 7–10 times longer than their counterparts in Figure 2, while the growth rates are 20–30 times slower. These numbers make the discrete modes relatively uninteresting for galactic applications, but they may yet prove important for accretion disks (see below).

### 4.2. Analytic Stability Criterion in the Limit of Long Horizontal Wavelengths

For the continuum modes the boundary in parameter and wavenumber space that separates unstable perturbations from stable ones corresponds to one where the square of the growth rate or frequency goes through zero (exchange of stabilities). The same principle does not apply to the discrete spectrum, because an unstable discrete mode can disappear by merging with the continuum before the physical conditions have changed sufficiently to stabilize it. Nevertheless, motivated by the numerical examples of the previous section, we find it useful to examine analytically the circumstances under which a discrete perturbation can grow in the limit of long horizontal wavelengths.

In the limit of $\xi, \eta \to 0$, with $\ell$ arbitrary, we have, at the margin of stability, $n = 0$, the following simplified forms for $E$ and $V_0$:

$$-E = -\frac{\eta^2}{\cos^2 \theta} \frac{1}{4}$$

$$V_0 = -\frac{\eta^2}{\cos^2 \theta} \frac{1}{4} + \frac{1}{4R},$$

where $\theta$ equals the angle between the horizontal wavevector and the equilibrium magnetic field,

$$\cos^2 \theta = \frac{\eta^2}{\zeta^2 + \eta^2}.$$ (4.3)

**Fig. 3.—**Growth rates for the discrete and continuum modes with radial wavenumber $\xi = 0$. The physical parameters read $(\alpha, \beta, \gamma, R) = (0.3, 0.0, 1.2, 3.5)$. Horizontal wavenumbers between 0.000 and 0.025 yield unstable disturbances for the discrete mode $l = 0$. Disturbances of all scales are stable for the higher order discrete modes, $l = 1, \ldots$, as they are for all the continuum modes with $\xi = 0$ (shaded region). The upper boundary for the shaded region corresponds to continuum modes with zero vertical wavenumber, $\xi = 0$.

**Fig. 4.—**Growth rates for the discrete and continuum modes with radial wavenumber $\xi = 0$. The physical parameters read $(\alpha, \beta, \gamma, R) = (0.2, 0.2, 1.3, 3.5)$. A range of instability exists for the two lowest discrete modes, $l = 0$ and $l = 1$, but the latter family has merged or is about ready to merge, throughout the entire range of stable and unstable azimuthal wavenumbers $\eta$, with the continuum family of modes (with $\xi = 0$) represented by the upper boundary of the shaded region.

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We can therefore set an upper bound on $\eta_0^2$ from the condition $E < 0$, and a lower bound from the requirement $\mathcal{N}_0 < 0$ (cf. eq. [3.13]) in the limit $n^2 \to 0+$. These two bounds yield

$$\frac{1}{4} > \frac{\eta_0^2}{\cos^2 \theta} > \frac{\ell(\ell+1)}{(2\ell+1)^2} \left[ 1 + \frac{1}{2R} - \frac{\ell(\ell+1)}{4R^2} \right].$$

(4.4)

The two inequalities have no physical range unless

$$\ell(\ell+1) < R.$$  

(4.5)

For application to the Galaxy in the solar neighborhood, where $R \approx 3.5$, we see that, at best, only two of the discrete modes ($\ell = 0$ and 1) can be relevant. For instability of long-wavelength $\ell = 0$ modes, we require $\frac{1}{8} > \eta_0^2/\cos^2 \theta > 0$; for instability of long-wavelength $\ell = 1$ modes, we require $\frac{1}{4} > \eta_0^2/\cos^2 \theta > 0.245$. In contrast, continuum modes become unstable whenever $\eta_0^2/\cos^2 \theta > \frac{1}{4}$. Discrete modes might prevail over continuum modes in the regions between spiral arms, where $\alpha$ and $\beta$ may be relatively small. Even there, the characteristic wavelength for instability is so long as to make dubious the validity of a local analysis. In any case, we expect the continuum modes to dominate in the more important regions inside spiral arms. In galactic applications, therefore, the discrete modes might be important only for defining the initial conditions for the onset of the more violent forms of Parker's instability that take place inside spiral arms.

The situation may be more favorable for applications to accretion disks. We have mentioned elsewhere that large $R$ mimics the case of linear gravity, $g_z \propto -z$. From equation (4.5), we also see that large $R$ yields more room to fit in a greater number of discrete modes. Nevertheless, the mode that has the dominant growth rate would still be $\ell = 0$, which has even parity (in $u_z$). We believe this result explains the robust tendency in the Parker instability calculations of Matsumoto et al. (1980) to find density condensations that alternate in location above and below the central plane. Presumably, if Matsumoto and colleagues were to repeat their calculations with a seed $\phi (z/2R)$ gravity law, with $R$ not too large, and $\eta_0^2$ not too small, they would find the tendency for even-parity structures to be much weaker.

In a related hydrodynamic stability analysis using linear gravity but ignoring magnetic fields, Nelson (1976a, b; Nelson & Matsuda 1980) obtained a spectrum with an infinite (countable) number of discrete modes, but no continuum. A similar result has been found in a recent nonmagnetic study by S. Lubow & J. Pringle (1992, private communication). If we add to these discussions the likelihood that accretion disks might exist close to the margin of convective stability (Lin & Papaloizou 1980), as well as have relatively weak magnetic fields (and, hopefully, sufficient electrical conductivity), then we would have the necessary ingredients (small positive $\eta_0^2$ and linear gravity) that would favor the onset of the fundamental discrete mode of Parker’s instability. Combined with the magnetorotational mechanism discussed by Balbus & Hawley (1991, 1992) and Hawley & Balbus (1991, 1992), this set of circumstances may give a viable physical foundation to the long-sought-for mechanisms of sustained dynamo action and anomalous viscosity in non–self-gravitating disks. Further discussion of these speculations lies beyond the scope of the present paper.

5. SUMMARY

Our finding of two distinct sets of disturbances, discrete modes spatially confined in $z$ and continuum modes which oscillate all the way to $\pm \infty$, results from using a gravitational field that reverses itself over a finite distance. Parker's constant-gravity assumption, with the sign of the gravitational field switching at the galactic plane, suppresses the discrete modes because the potential well in the analogous quantum mechanical problem has no width. On the other hand, using a gravity proportional to $z$ suppresses the continuous spectrum, rather like the completely bounded harmonic-oscillator problem in quantum mechanics. In all cases, the modes split into sequences of even and odd parity. Unstable modes of even parity allow growing motions that cross the midplane, a feature that may be observationally useful for distinguishing the action of Parker's instability in comparison with other contenders (but not all others).

The presence of both continuum and discrete modes, and the allowance of both kinds of parities, in the general case do not depend on the equation of state of the gas or the particular analytic form chosen for $g_z$. These features will generally arise whenever $g_z$ is an odd function of $z$ that smoothly goes through zero at the midplane and does not acquire indefinitely large values as $|z| \to \infty$.

We believe these basic points will survive more complete analyses that include the effects of self-gravity of the gas as well as differential rotation in the horizontal directions. Astronomers have just begun to explore the full set of three-dimensional instabilities that can afflict a magnetized, differentially rotating, and possibly self-gravitating layer of gas. The possibilities are rich, perhaps even as rich as the wide variety of observational phenomena that have been already ascribed to such objects.

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