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AMBIPOLAR DIFFUSION IN SELF-GRAVITATING ISOTHERMAL LAYERS

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ABSTRACT

We formulate and solve the problem of the drift of magnetic field and ions embedded in a self-gravitating layer of neutral isothermal gas under the assumptions of quasi-magnetohydrostatic equilibrium and local ionization equilibrium. When Lagrangian coordinates referred to the neutral gas are introduced, we find that the problem can be reduced to a nonlinear diffusion equation for the magnetic field, whose dimensionless form involves no parameters other than those introduced by the initial values. A few numerical solutions for the initial-value problem are presented, all of which converge asymptotically in time to a shape-invariant form. The shape-invariant solution for our problem is then sought and found. In the shape-invariant solution, the magnetic field at each surface-density point in the neutral fluid decays as the inverse square root of the elapsed time. As a function of the ratio of the initial magnetic to neutral-gas pressure in a natural family of cases, we give explicit estimates for the amount of time that needs to pass before the shape-invariant solution becomes a good approximation for the actual behavior. We interpret our results physically, relate them to previous work, comment on possible implications for the problem of star formation, and speculate on future extensions into more realistic problems involving two or three spatial dimensions.

Subject headings: hydromagnetics — nebulae: general

I. INTRODUCTION

More than a quarter of a century ago, Mestel and Spitzer (1956) proposed a process, ambipolar diffusion, by which interstellar magnetic field, frozen to the charged particles, may nevertheless drift relative to the neutrals in a lightly ionized medium. Maxwell stresses that act only on the ions (and electrons) of the system drive them through the neutrals (which may or may not be at rest) at a speed which quickly reaches the terminal velocity set by the exchange of momentum in ion-neutral collisions. Appreciable magnetic flux may be lost in this manner from regions of low-fractional ionization (for a brief review, see Spitzer 1978, pp. 293–296). It is not yet clear if the low values of magnetic fields inferred for many regions of interstellar space (Heiles 1976) already demands the operation of this mechanism, but there seems to be little doubt that it prevails at some level in the process of star formation from the dense cores of molecular clouds (Mestel 1977; Mouschovias 1981; Nakano 1981). Star formation is the nominal concern of the current paper, but ambipolar diffusion is also the basic ingredient in current discussions of magnetic precursors in shocked molecular clouds (Mullan 1971; Draine 1980; Chernoff, Hollenbach, and McKee 1982; Draine, Roberge, and Dalgarno 1982).

In contrast to the many detailed calculations of the rates for the relevant microscopic processes (e.g., Nakano and Tademaru 1972; Elmegreen 1979), the rate of the macroscopic leakage of the field has either received only order-of-magnitude estimates or has involved heavy numerical computations (e.g., Nakano 1979, 1982; Black and Scott 1982). A notable exception is Mouschovias and Paleologou (1981), who analytically solved for the escape of ions and magnetic field from a uniform slab of neutrals. Integrating over an approximate description of the spatial structures of the magnetic field and the ion density and velocity, they were able to reduce the governing equations to ordinary differential equations that exhibit both the transient rise of the drift velocity to its terminal value, as well as the asymptotic power-law decay of the magnetic field at large times (see Fig. 1 of Mouschovias and Paleologou 1981).

This paper extends the class of solutions accessible to study without extensive numerical computations. We restrict our system to be stratified in plane-parallel layers; we include the important effect of the self-gravitation of the neutral component; and we update the force balance and vertical structure of the neutral gas as the magnetic field and ions leak from it. Our method allows us to explore parameter space rather thoroughly and to derive solutions which are easily interpretable and have elegant mathematical properties. Unfortunately, the problem that we have posed—a plane-parallel gaseous layer, extending infinitely in the horizontal directions, and supported, at least in part, against its own weight by horizontal magnetic fields—is unstable against the magnetic Rayleigh-Taylor instability (Parker 1966). Since this instability generally occurs on a dynamical time scale that is appreciably shorter than the plasma drift time scale, the initial state that we have adopted cannot arise in nature, and our solutions should not be directly compared to any known astronomical object. This difficulty can be avoided in a three-dimensional cloud, for the Parker instability is suppressed if the length of the object along the direction of the magnetic field is smaller than the critical scale of the instability. Despite these *a priori* objections to the plane-parallel problem, our efforts may have long-range utility since (*a*) they provide a useful set of test cases for checking complex computer codes, and (*b*) they suggest novel techniques for treating the more realistic cases.

II. BASIC EQUATIONS AND ASSUMPTIONS

Consider the development in space z and time t of a partially ionized and magnetized medium which is stratified in plane-parallel layers with magnetic field lines that are straight and lie parallel to surfaces of constant z. We denote the density, fluid velocity (in the z-direction), and mean molecular mass of the neutrals by ρ, u, and m; the corresponding quantities for the ions are given by ρ_i, u_i, and m_i. In molecular clouds, $m \approx 2.3\ m_{\rm H}$ and $m_i \approx 30\ m_{\rm H}$, where $m_{\rm H}$ is the mass of the hydrogen atom (cf. de Jong, Dalgarno, and Boland 1980). The magnetic field B is assumed to be frozen in the fluid of charged particles, and we approximate the electrons to have negligible inertia and to move with the ions, keeping the overall fluid electrically neutral. We also assume that the heating and cooling is such as to keep the neutral gas effectively isothermal so that its pressure is given by $a^2\rho$, where the isothermal speed of sound a is a constant. If, furthermore, the evolution of the system is slow in comparison with the ionization and recombination time scales, we may take the local ion density to be given by ionization equilibrium. For dense molecular clouds, the ionization rate per unit volume by cosmic rays is proportional to the first power of the neutral gas density while recombination is proportional to the second power of the ion gas density. When the two rates are in balance, we have

$$\rho_i = C\rho^{1/2}\ , \tag{1}$$

where the proportionality constant C can be obtained from observations (e.g., Guelin, Langer, and Wilson 1982) or theory (e.g., Elmegreen 1979). Roughly, $C \sim 3 \times 10^{-16}\ {\rm g}^{1/2}\ {\rm cm}^{-3/2}$. For densities of interest to the dense cores of molecular clouds, $\rho \sim 10^{-18}\ {\rm g\ cm^{-3}}$, we have $\rho_i \ll \rho$, by over six orders of magnitude. This has the following important consequences. First, we may ignore the self-gravity of the ions in comparison with that of the neutrals. Second, if the magnetic field is to be dynamically interesting, the magnetic pressure $B^2/8\pi$ must be at least comparable to the neutral gas pressure $a^2\rho$. But in the absence of violent disturbances (e.g., strong shocks), the ions are likely to have the same temperature as the neutrals, and this then means that the ion pressure $a_i{}^2\rho_i$ must be completely negligible in comparison with the magnetic pressure $B^2/8\pi$.

With the above approximation, we may write the dynamical equations in their Eulerian forms as

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial z} = -\frac{1}{\rho_i}\frac{\partial}{\partial z}\left(\frac{B^2}{8\pi}\right) + g - \rho\gamma(u_i - u) - \zeta\frac{\rho}{\rho_i}(u_i - u)\ , \tag{2}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0\ , \tag{3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = g - \frac{a^2}{\rho}\frac{\partial \rho}{\partial z} - \rho_i\gamma(u - u_i) - \zeta(u - u_i)\ . \tag{4}$$

Notice that in the present formulation, the equation of continuity for the ions is replaced by the assumption of instantaneous ionization equilibrium, equation (1). In equations (2) and (4), g is the gravitational field, and γ is the drag coefficient arising from momentum exchange in ion-neutral collisions. For low drift speeds, it may be taken to be $\gamma \approx 3.5 \times 10^{13}\ {\rm cm^3\ (g\text{-}s)^{-1}}$ (cf. eq. [12] of Draine, Roberge, and Dalgarno 1982). The last terms in equations (2) and (4) enter because each cosmic-ray ionization at a rate ζ per neutral particle is accompanied statistically by a recombination of ion and electron whose average velocity differs by the quantity $(u_i - u)$. Since $\zeta \sim 10^{-17}\ {\rm s^{-1}}$, neutrals are ionized about six orders of magnitude less frequently than they collide with existing ions in the dense cores of molecular clouds so the contribution to the momentum exchange between neutrals and ions due to ionization and recombination is much less than due to ion-neutral collisions. Henceforth, we drop the terms proportional to ζ in equations (2) and (4).

Equations (1)–(4) break down for $\rho > 10^{-16}\ {\rm g\ cm^{-3}}$. First, collisions between charged dust grains and neutrals dominate over the momentum transfer due to ion-neutral interactions at very high densities, tending to decrease the slippage of the magnetic field with respect to the neutrals (Elmegreen 1979; Draine 1980; Nakano and Umebayashi 1980). Second, because of shielding, cosmic rays become less important than natural radioactivity (primarily ^{40}K) as a source of ionization at high column densities (Nakano and Tademaru 1972). Third, at high densities, recombination of ions on grains changes equation (1) to $\rho_i = $ const (Umebayashi and Nakano 1980). Nevertheless, the major flux loss in realistic situations may occur before we reach densities greater than $10^{-16}\ {\rm g\ cm^{-3}}$.

To complete the set of equations (1)–(4), we need to include the equations for the magnetic and gravitational fields:

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial z}(Bu_i) = 0\ , \tag{5}$$

$$\frac{\partial g}{\partial z} = -4\pi G\rho\ , \tag{6}$$

where in the right-hand side of equation (6), we have ignored ρ_i with respect to ρ.

It is convenient to introduce the drift velocity,

$$v \equiv u_i - u \,, \tag{7}$$

where, consistent with the assumption that ρ_i is very small, equation (2) may be approximated by setting v equal to the terminal velocity forced by the gradient of the magnetic pressure:

$$v = -\frac{1}{\gamma \rho_i \rho} \frac{\partial}{\partial z}\left(\frac{B^2}{8\pi}\right). \tag{8}$$

Because we have ignored the gravitational field in expression (8), we have allowed the average ion ultimately to spread to infinity instead of merely reaching a very large scale height (comparable to the size of the Galaxy for typical numbers). If we denote the substantial derivative with respect to the *neutrals* by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial z}\,, \tag{9}$$

we may now write equations (3)–(5) as

$$\frac{D\rho}{Dt} = -\rho \frac{\partial u}{\partial z}\,, \tag{10}$$

$$\frac{Du}{Dt} = g - \frac{a^2}{\rho}\frac{\partial \rho}{\partial z} - \frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{B^2}{8\pi}\right), \tag{11}$$

$$\frac{D}{Dt}\left(\frac{B}{\rho}\right) = -\frac{1}{\rho}\frac{\partial}{\partial z}(Bv)\,. \tag{12}$$

The gradient of the magnetic pressure enters in the equation of motion (11) for the neutrals only because we have used the frictional coupling formula (8).

A further simplification is possible if we introduce Lagrangian coordinates, i.e., let us transform $(z, t) \rightarrow (\sigma, t)$, where σ is the surface density between the midplane and $z > 0$:

$$\sigma \equiv \int_0^z \rho(z', t)dz'\,. \tag{13}$$

(Notice that the usually defined surface density from $-z$ to $+z$ has twice the value of σ.) In the transformation from Eulerian to Lagrangian descriptions, we have

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_\sigma\,, \quad \left(\frac{\partial}{\partial z}\right)_t = \rho\left(\frac{\partial}{\partial \sigma}\right)_t\,, \quad u = \left(\frac{\partial z}{\partial t}\right)_\sigma\,.$$

With the above, equation (8) becomes

$$v = -\frac{1}{\gamma \rho_i}\frac{\partial}{\partial \sigma}\left(\frac{B^2}{8\pi}\right),$$

where from here on, partial derivatives are taken either with constant t or constant σ. The field equation (6) for g can be integrated to give Gauss's law:

$$g = -4\pi G\sigma\,, \tag{14}$$

while the field equation (12) for B now takes the form:

$$\frac{\partial}{\partial t}\left(\frac{B}{\rho}\right) = \frac{1}{\gamma}\frac{\partial}{\partial \sigma}\left(\frac{B^2}{4\pi\rho_i}\frac{\partial B}{\partial \sigma}\right). \tag{15}$$

The equation of continuity (10) for the neutrals is replaced by the differential form of equation (13),

$$\frac{\partial z}{\partial \sigma} = \frac{1}{\rho}\,, \tag{16}$$

while the equation of motion (11) for the neutrals becomes

$$\frac{\partial^2 z}{\partial t^2} = -4\pi G\sigma - a^2\frac{\partial \rho}{\partial \sigma} - \frac{\partial}{\partial \sigma}\left(\frac{B^2}{8\pi}\right). \tag{17}$$

When ρ_i is given by equation (1), equations (15), (16), and (17) form our basic set to solve for B, ρ, and z as functions of σ and t.

We are interested in problems where the initial state is one of magnetohydrostatic equilibrium and the time evolution is solely due to ambipolar diffusion. To be sure, as the magnetic field diffuses out of the neutral gas, this gas will settle gravitationally toward the midplane until, asymptotically in time, it is supported against its weight only by its own thermal pressure. But the speeds at which it will do this can be justified *a posteriori* to be small, and since an isothermal layer can never go into gravitational collapse as long as it is confined to variations only in the vertical direction (Spitzer 1968, pp. 45–46), it must be an acceptable approximation to ignore the inertial term on the left-hand side of equation (17). In this approximation of quasi-magnetohydrostatic equilibrium, we may integrate the force balance to obtain

$$\frac{B^2}{8\pi} + a^2\rho = 2\pi G({\sigma_\infty}^2 - \sigma^2)\,. \tag{18}$$

The integration constant σ_∞ may be identified as the value of σ at $z = \infty$, where B and ρ are zero. Equation (18) states that, at every layer, the total pressure is equal to the weight of the material on top. The algebraic relation introduced between B and ρ at each σ decouples equation (15) from equation (16), so that equation (15) may now be regarded as a nonlinear diffusion equation for B where the diffusion coefficient is to be calculated with the help of equations (1) and (18). In the next section, we nondimensionalize and present numerical solutions for this nonlinear diffusion problem.

III. DIMENSIONLESS EQUATIONS AND SOLUTIONS

Introduce the dimensionless surface density μ, volume density p, magnetic field b, vertical coordinate y, and time τ through the definitions:

$$\sigma \equiv \sigma_\infty \mu\,, \tag{19a}$$

$$\rho \equiv \frac{2\pi G{\sigma_\infty}^2}{a^2}\,p\,, \tag{19b}$$

$$B \equiv 4\pi G^{1/2}\sigma_\infty b\,, \tag{19c}$$

$$z \equiv \frac{a^2}{2\pi G\sigma_\infty}\,y\,, \tag{19d}$$

$$t \equiv \left[\frac{\gamma C}{2(2\pi G)^{1/2}}\right]\left[\frac{a}{2\pi G\sigma_\infty}\right]\tau\,. \tag{19e}$$

With these definitions, equations (15), (18), and (16) become

$$\frac{\partial}{\partial \tau}\left(\frac{b}{p}\right) = \frac{\partial}{\partial \mu}\left(\frac{b^2}{p^{1/2}}\frac{\partial b}{\partial \mu}\right), \tag{20}$$

$$b^2 + p = 1 - \mu^2 , \tag{21}$$

$$\frac{\partial y}{\partial \mu} = \frac{1}{p} . \tag{22}$$

Equation (20) is to be solved under the boundary conditions

$$\frac{\partial b}{\partial \mu} = 0 \text{ at } \mu = 0 \quad \text{and} \quad b = 0 \text{ at } \mu = 1 . \tag{23}$$

The first condition makes $\boldsymbol{B}$ possess reflection symmetry with respect to the midplane; the second makes $\boldsymbol{B}$ vanish at $z = \infty$. Given initial values, we can find b and p for all later times from equations (20) and (21); then equation (22) can be integrated subject to the midplane condition

$$y = 0 \text{ at } \mu = 0 \tag{24}$$

to obtain the transformation back to an Eulerian description.

Notice that once we have scaled in accordance with equations (19), the resulting dimensionless equations (20)–(22) contain no nondimensional parameters. In the current problem, therefore, nondimensional parameters can enter only as part of the initial data. For a choice where b^2 and p are initially not very different from each other (roughly equipartition conditions), we can expect the product of the two brackets in equation (19e) to be a characteristic (drift) time scale. The quantity represented by the second bracket is the sound-crossing time scale of the neutral gas in its final equilibrium state, since it is the scale height $a^2/2\pi G\sigma_\infty$ (cf. eq. [19d]) divided by the isothermal sound speed a. The quantity in the first bracket

$$\frac{\gamma C}{2(2\pi G)^{1/2}}$$

represents, therefore, the number of sound-crossing time scales that is contained in the drift time scale. For $\gamma C = 1 \times 10^{-2}$ cm$^{3/2}$ g$^{-1/2}$ s^{-1}, this multiple equals 8. Since the square of 8 is large in comparison with unity, we will usually be justified in our assumption of quasi-magnetohydrostatic equilibrium. Minor corrections can be made by perturbation theory or by carrying along numerically the acceleration of the neutrals in an implicit fashion (using past solutions to difference z twice in time), but we forego such corrections here. Since the sound-crossing time scale in the cores of molecular clouds is typically 10^5 yr (Myers and Benson 1983), the drift of magnetic fields out of molecular cores into their envelopes (see Fig. 3 below) takes on the order of 10^7 yr, which is the extent to which star formation in T associations is not coeval (Cohen and Kuhi 1979).

Since equation (20) represents a (nonlinear) diffusion equation which is subject to the homogeneous boundary conditions (23), we may expect that all solutions for b will eventually tend toward zero for every μ (everywhere within the neutral gas). Thus, solving equation (22) subject to equations (21) and (24) when $b = 0$, we can predict that all solutions must eventually become the *final* state for a self-gravitating isothermal layer (Spitzer 1942):

$$p = 1 - \mu^2 = \operatorname{sech}^2 y \text{ at } \tau = \infty \text{ with } \mu = \tanh y . \tag{25}$$

A natural family of *initial* states is generated by assuming that the initial ratio of magnetic to gas pressure is everywhere a constant, α_0, i.e., $b^2/p = \alpha_0$ at $\tau = 0$. To satisfy equations (21), (22), and (24), this state must satisfy

$$p = \frac{1}{1+\alpha_0}(1-\mu^2) = \frac{1}{1+\alpha_0}\operatorname{sech}^2\left(\frac{y}{1+\alpha_0}\right), \tag{26a}$$

$$b = \left(\frac{\alpha_0}{1+\alpha_0}\right)^{1/2}(1-\mu^2)^{1/2} = \left(\frac{\alpha_0}{1+\alpha_0}\right)^{1/2}\operatorname{sech}\left(\frac{y}{1+\alpha_0}\right), \tag{26b}$$

$$\mu = \tanh\left(\frac{y}{1+\alpha_0}\right), \tag{26c}$$

at $\tau = 0$. To see how state (26) evolves to state (25) requires us numerically to solve equations (20)–(22) using equations (26) as initial data.

The numerical solution of equations (20)–(22) is easily effected by finite difference techniques. The solutions found in this paper were generated by an explicit method which differenced the right-hand side of equation (20) on 41 equally spaced (so that $\Delta\mu = 1/40$) grid points on the interval $\mu = 0$–1, and advanced in time in steps of $\Delta\tau = \Delta\mu^2$ (for numerical stability). The integrations were carried out to time $\tau = 20$ (involving $20 \times 1600 = 320{,}000$ individual time steps), a chore that took about 10 minutes on the VAX at the Institute for Advanced Study. Except on the next-to-last grid point, solutions generated with half the number of grid points differed typically by less than the plotting resolution of the figures presented in this paper.

Figure 1*a* presents graphically the solution for $b(\mu, \tau)$ for the initial condition $\alpha_0 = 1$. Figures 1*b* and 1*c* give the quantities $y(\mu, \tau)$ and $p(y, \tau)$ for the same case. Notice how as the magnetic field leaks from the neutral gas (Fig. 1*a*), the neutral gas gradually settles vertically (Fig. 1*b*), and the volume density of the neutrals shifts in profile from equation (26a) to equation (25) (Fig. 1*c*).

Figures 2*a*–2*c* present the same quantities for the initial condition $\alpha_0 = 10$. The large-time behavior is especially interesting. The convergence of the density profile independent of the initial condition to the form (25) was, of course, anticipated by our previous discussion. What was not anticipated is the convergence of the asymptotic behavior of the magnetic field b that can be seen on comparing Figures 1*a* and 2*a*. To show that this convergence is not restricted to the family of

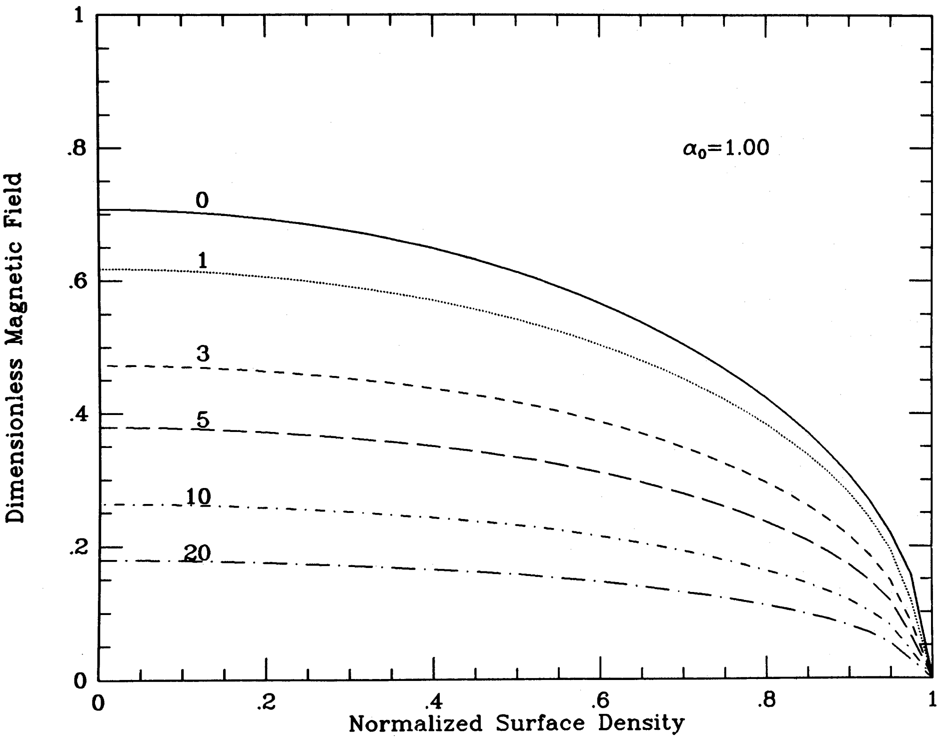

FIG. 1*a*

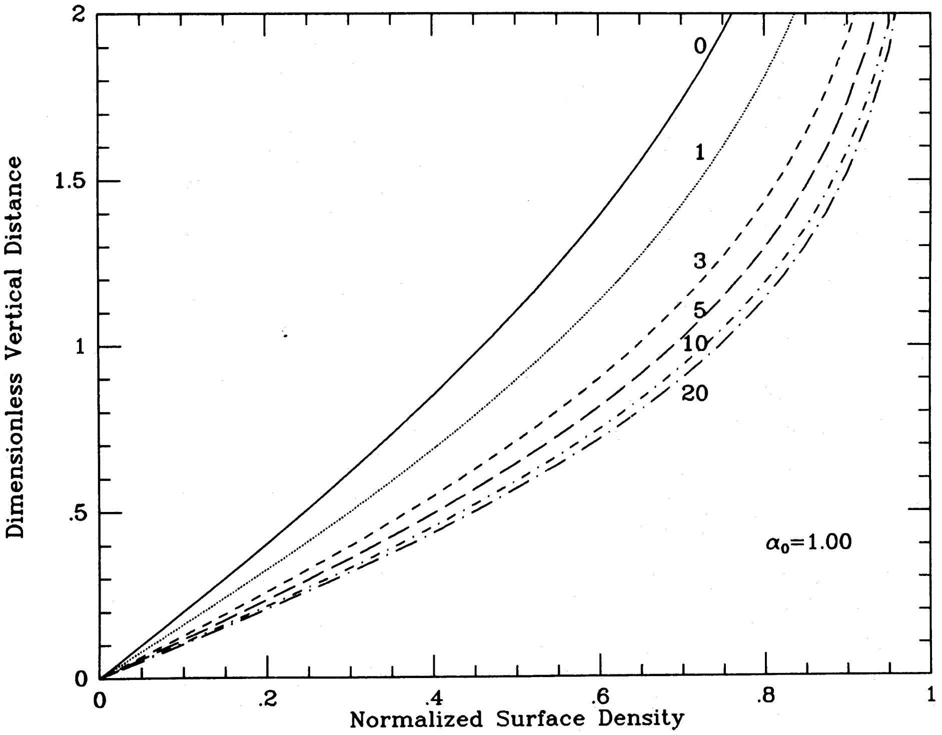

FIG. 1*b*

FIG. 1.—Ambipolar diffusion at dimensionless times τ equal to 0, 1, 3, 5, 10, and 20, for the case when the ratio α_0 of the initial magnetic to gas pressure equals 1. (*a*) The dimensionless magnetic field b plotted against the normalized surface density μ at various times. (*b*) The dimensionless vertical coordinate y plotted against the normalized surface density μ at various times. The dimensionless velocity of the neutral gas's motion can be obtained by differencing y at a given μ in time τ. The dimensional velocity is smaller than free-fall values roughly by the additional factor $2(2\pi G)^{1/2}/\gamma C$. (*c*) The dimensionless neutral gas density p plotted against the dimensionless vertical coordinate y at various times.

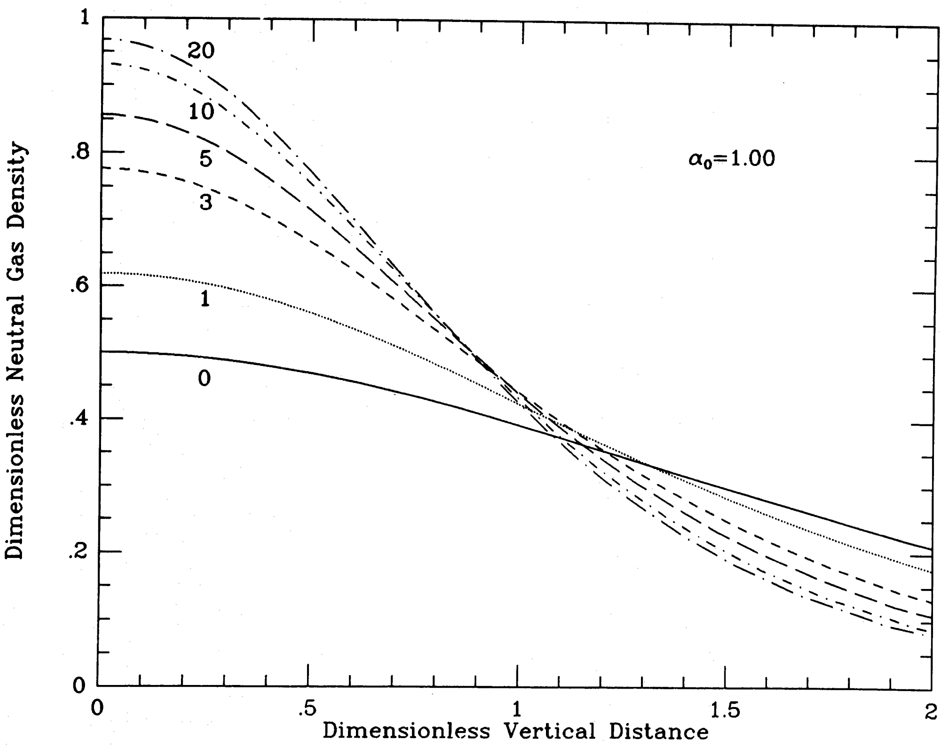

FIG. 1c

initial-value problems (26), consider Figures 3*a* and 3*b*. The initial magnetic field for this calculation was chosen to be

$$b = \mu(1 - \mu^2)^{1/2} , \tag{27a}$$

implying from equation (22),

$$p = (1 - \mu^2)^2 , \tag{27b}$$

which allows the integration of equation (22) as

$$y = \frac{1}{2}\left[\operatorname{arctanh} \mu + \frac{\mu}{(1 - \mu^2)}\right] . \tag{27c}$$

The initial field strength (27a) has a nonvanishing first derivative at $\mu = 0$, but after one time step ($\tau = 1/1600$) this is ironed out by the imposition of the boundary condition (23). The important point is that the early evolution of this case is very different from the previous two displayed examples (compare Figs. 1*a*, 2*a*, and 3*a*). The off-center maximum of the magnetic field causes the field near the midplane to *increase* before it later decreases. This behavior manifests itself in Figure 3*b* in the midplane volume density *decreasing* at first when magnetic field drifts into this region to help support the system against its own gravity. But inevitably, the midplane density increases again after the magnetic pressure gradient everywhere takes the same sign and the ions migrate relentlessly from the neutral gas. By dimensionless time $\tau = 3$ onwards, the magnetic field (Fig. 3*a*) has acquired a profile reminiscent of the forms in Figures 1*a* and 1*b*.

IV. SHAPE-INVARIANT SOLUTION

The closing comments of the previous section suggest strongly that, independent of the initial conditions, all of the solutions for the magnetic field acquire, asymptotically in time, a shape-invariant form (like a decaying mode). Motivated by this comment, we look for solutions of equation (20) which have separable dependences on μ and τ:

$$b(\mu, \tau) = Z(\mu)T(\tau) . \tag{28}$$

The substitution of equation (28) into equation (20) results in the expression,

$$\frac{1}{T^3}\frac{dT}{d\tau} = \frac{p}{Z}\frac{d}{d\mu}\left(\frac{Z^2}{p^{1/2}}\frac{dZ}{d\mu}\right) , \tag{29}$$

where, according to equation (21), p is approximately given by

$$p = 1 - \mu^2 , \tag{30}$$

when the magnetic pressure b^2 has become small in comparison with the neutral gas pressure p.

The left-hand side of equation (29) is a function only of τ; the right-hand side, only of μ. For the equation to be valid, both sides must equal a constant, say, $-k$. It is easy to show that a given choice of k scales the function T by $k^{-1/2}$, and the function Z by $k^{1/2}$, leaving the product TZ unchanged. Since it is only the product which enters in the physical variable, the magnetic field (28), we may, without loss of generality,

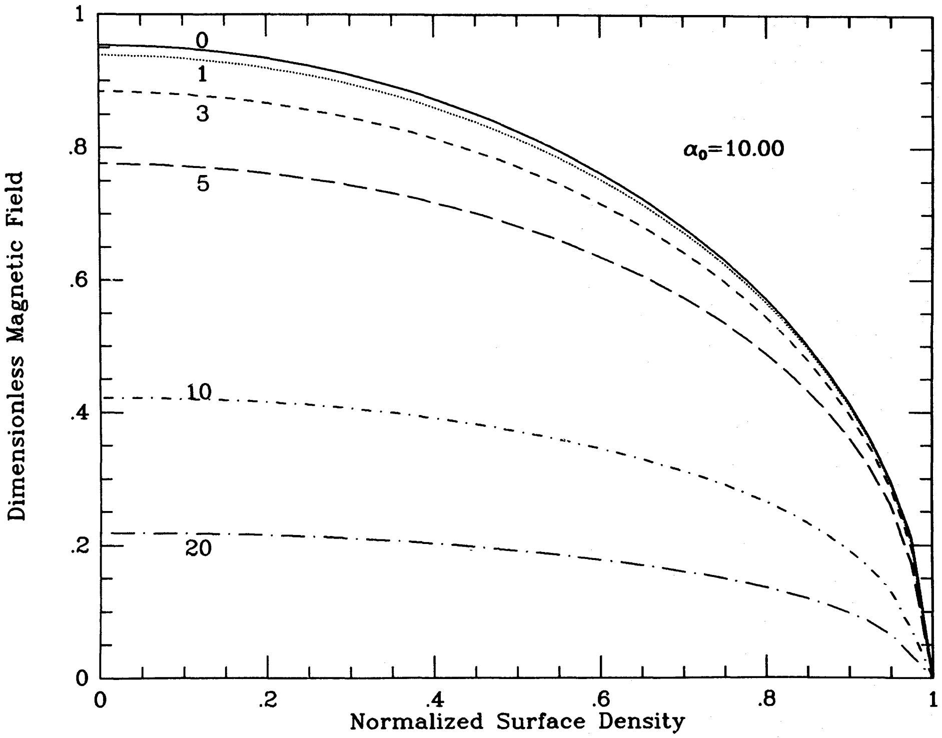

FIG. 2a

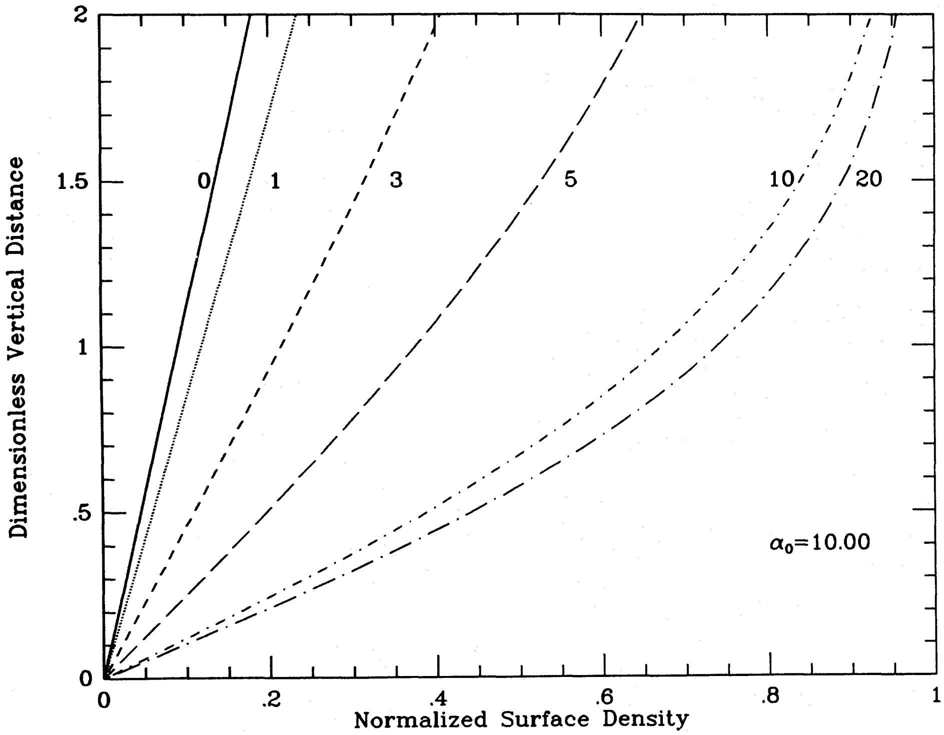

FIG. 2b

FIG. 2.—Ambipolar diffusion at dimensionless times τ equal to 0, 1, 3, 5, 10, and 20, for the case when the ratio α_0 of the initial magnetic to gas pressure equals 10. (a) The dimensionless magnetic field b plotted against the normalized surface density μ at various times. (b) The dimensionless vertical coordinate y plotted against the normalized surface density μ at various times. The dimensionless velocity of the neutral gas's motion can be obtained by differencing y at a given μ in time τ. The dimensional velocity is smaller than free-fall values roughly by the additional factor $2(2\pi G)^{1/2}/\gamma C$. (c) The dimensionless neutral gas density p plotted against the dimensionless vertical coordinate y at various times.

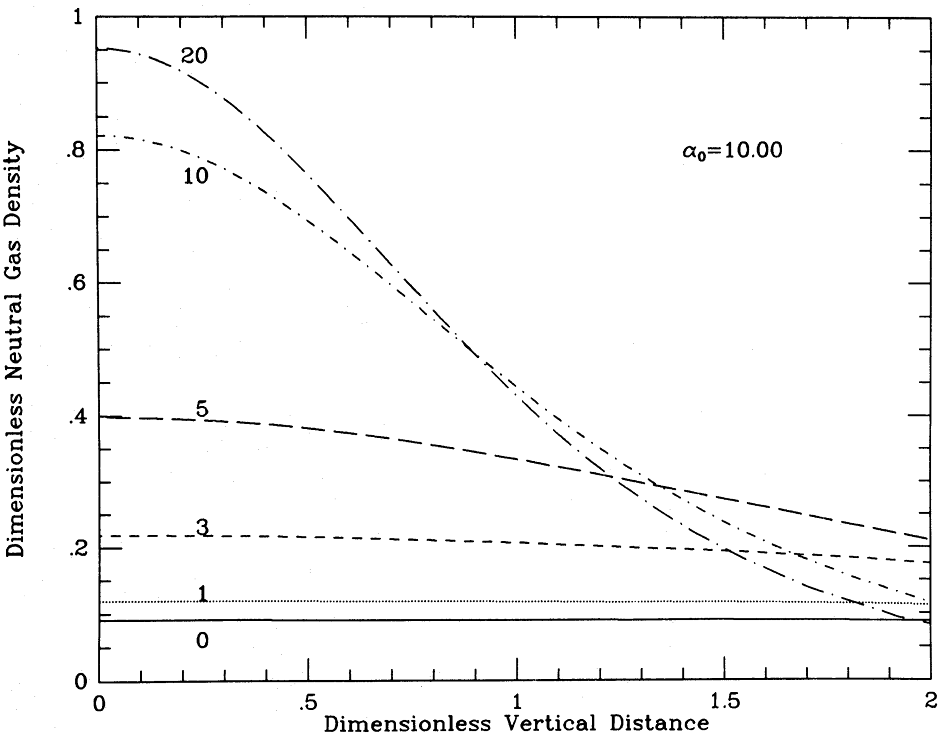

FIG. 2c

set both sides of equation (29) equal to -1. The resulting ordinary differential equation for T can be integrated analytically to give

$$T(\tau) = \frac{1}{[2(\tau - \tau_0)]^{1/2}}\,, \tag{31}$$

where τ_0 is an integration constant. The ordinary differential equation for Z reads

$$\frac{d}{d\mu}\left(\frac{Z^2}{p^{1/2}}\frac{dZ}{d\mu}\right) + \frac{Z}{p} = 0\,, \tag{32}$$

which is to be solved under the two-point boundary conditions,

$$\frac{dZ}{d\mu} = 0 \text{ at } \mu = 0 \quad \text{and} \quad Z = 0 \text{ at } \mu = 1\,. \tag{33}$$

The integration of equation (32) must be performed numerically. Since $\mu = 1$ is a singular point of the equation, we found it useful to develop Z as the following series near $\mu = 1$,

$$Z = Ap^{1/2} - \frac{1}{4A}p + \left(\frac{A}{10} - \frac{3}{80A^3}\right)p^{3/2} + \cdots\,, \tag{34}$$

where p is given by equation (30). Clearly, the above series satisfies the constraint (33) at $\mu = 1$; the constant A is to be chosen so that the other boundary condition at $\mu = 0$ can also be met. It is a simple matter to step slightly off $\mu = 1$ with the help of equation (34), continue the solution by numerical integration of equation (32), and use a Newton-Raphson technique to find the value of A which makes $dZ/d\mu$ zero at $\mu = 0$. Figure 4 presents graphically the solution for the desired Z.

To determine the integration constant τ_0 in equation (31), we used the following technique. For the family of initial-value problems (26), we defined an artificial τ_0 such that the shape-invariant solution (28) gave exactly the correct midplane value $b(0, \tau)$ at each τ for which we had a finite-difference solution. For given α_0, we then plotted the artificial τ_0 against $1/\tau$. This curve could be extrapolated to $1/\tau = 0$ (the asymptotic state of interest), and we considered the intercept to define τ_0 for that choice of α_0. Figure 5 shows a smooth fit through the data points of τ_0 versus α_0 obtained in this fashion. The time constant τ_0 in this graph represents the value which τ must exceed significantly, for given ratio α_0 of initial magnetic to gas pressure, before the true solution is reasonably represented by the shape-invariant solution. To fix ideas, consider the case $\alpha_0 = 1$, for which $\tau_0 = 2.8$. Plotted in Figure 6 is the product TZ (cf. eq. [28]) for $\tau = 5$, 10, and 20 when $\tau_0 = 2.8$. If we compare Figure 6 to Figure 1*a*, we find that the shape-invariant solution overrepresents the actual magnetic field $b(\mu, \tau)$ at $\tau = 5$, that it only slightly exceeds the correct solution by $\tau = 10$, and that, except for the very tail, it is indistinguishable from the true solution by $\tau = 20$. This is not atypical; other cases we have tried suggest that the shape-invariant solution gives a very good approximation to the actual solution by 2 or $3\tau_0$.

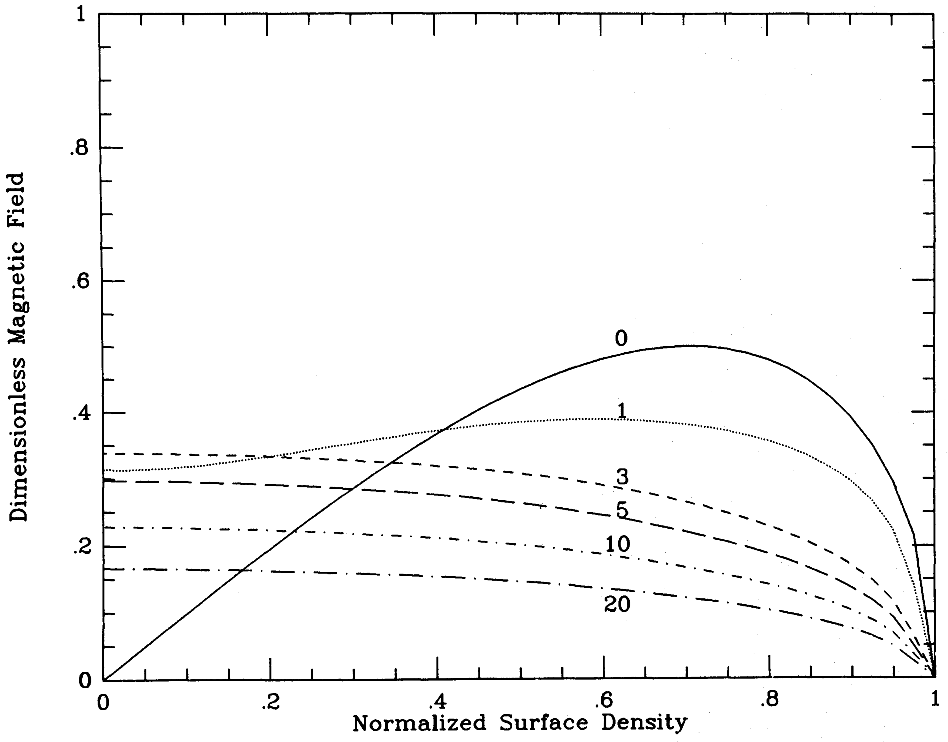

FIG. 3a

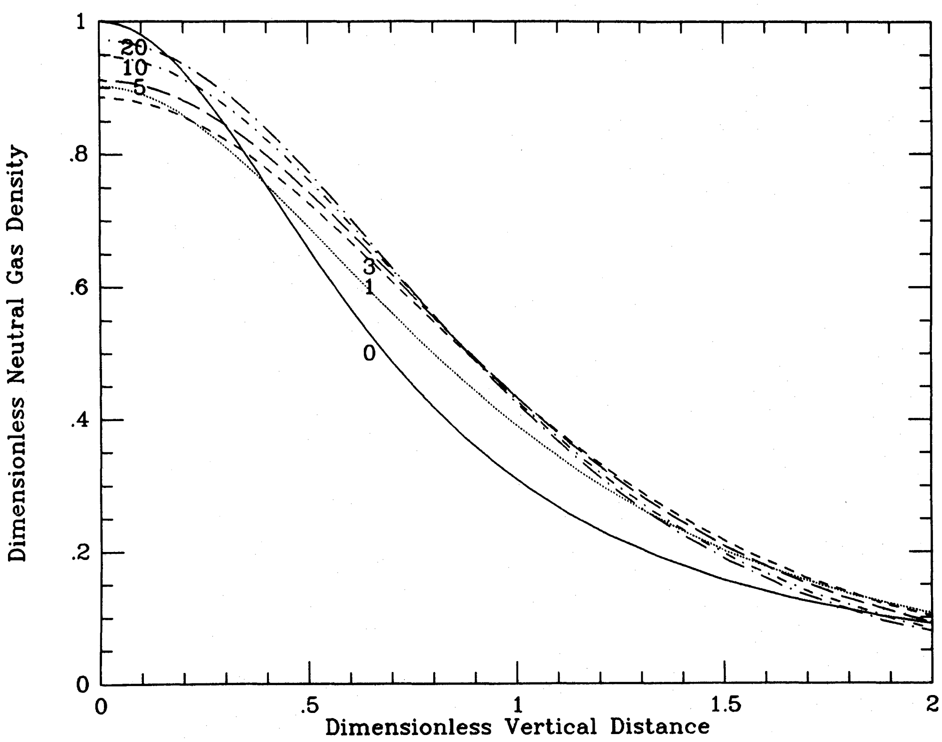

FIG. 3b

FIG. 3.—Ambipolar diffusion at dimensionless times τ equal to 0, 1, 3, 5, 10, and 20, for the case whose initial conditions are described by equations (27). (*a*) The dimensionless magnetic field b plotted against the normalized surface density μ at various times. (*b*) The dimensionless neutral gas density p plotted against the dimensionless vertical coordinate y at various times.

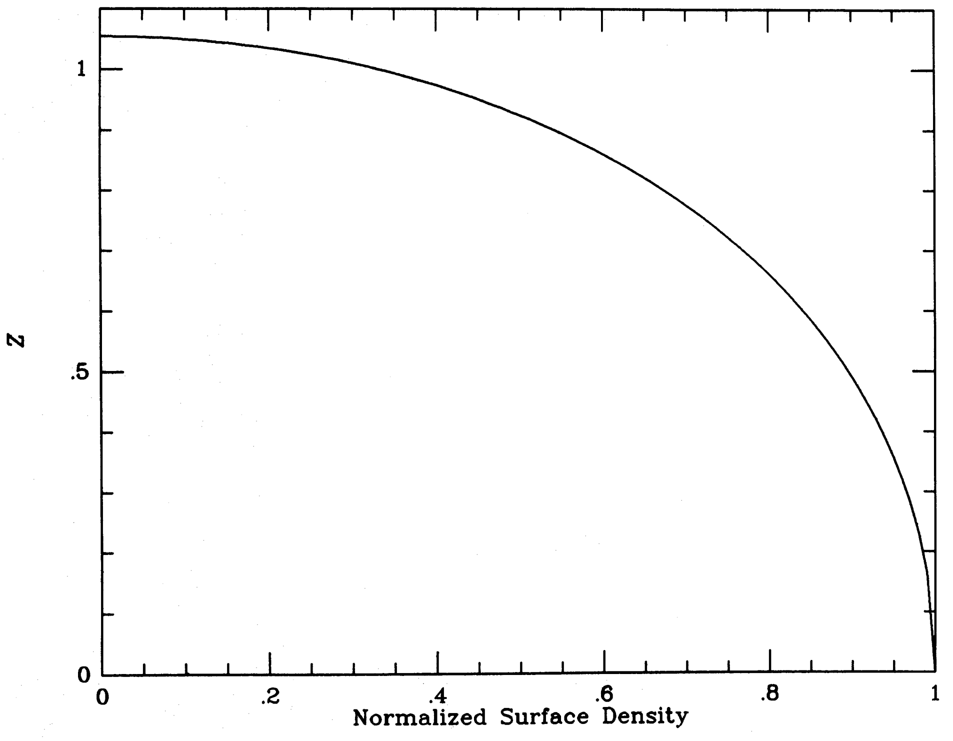

FIG. 4.—The spatial function $Z(\mu)$ defined by eqs. (32) and (33)

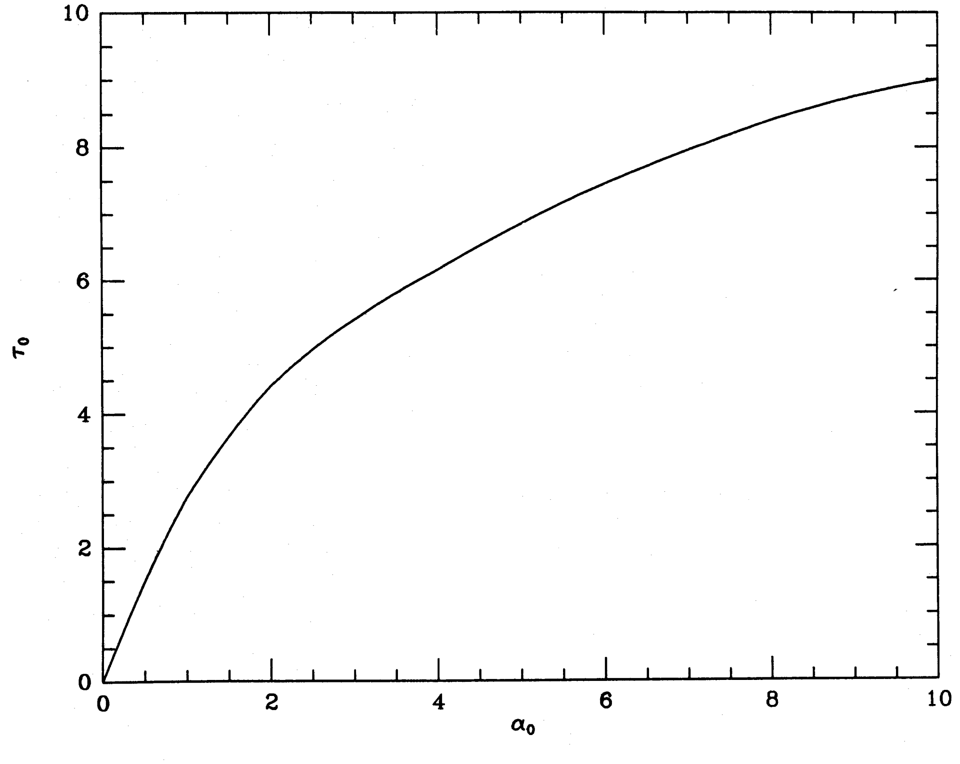

FIG. 5.—The optimum choice for the integration constant τ_0 in eq. (31) as a function of the ratio of initial magnetic to gas pressure α_0 (see eqs. [26a]–[26c]).

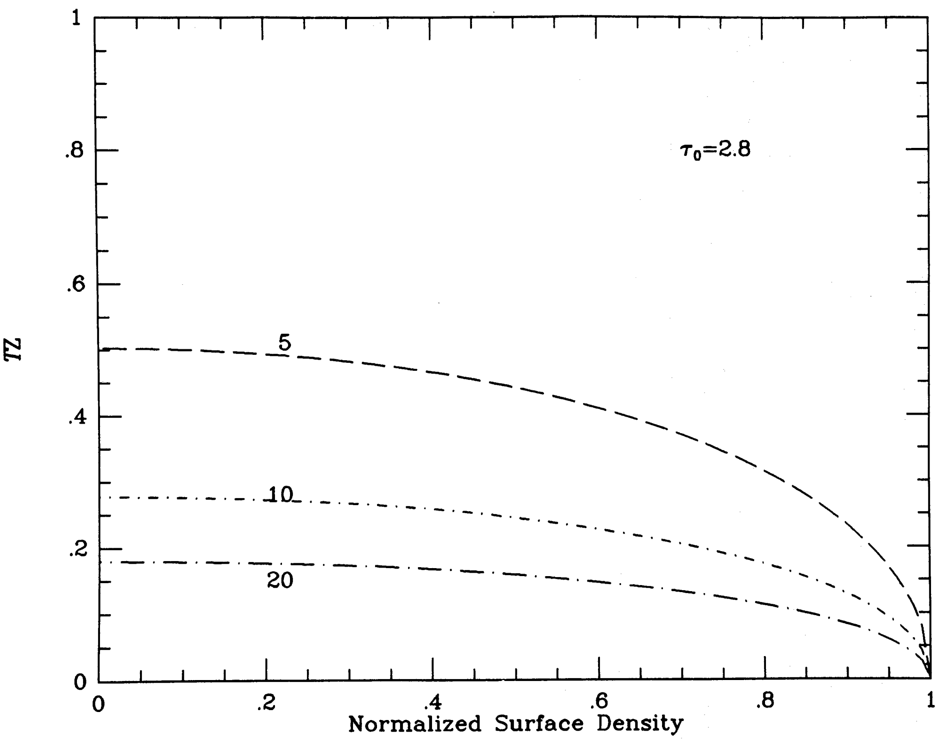

FIG. 6.—The shape-invariant solution $b = TZ$ at dimensionless times τ equal to 5, 10, and 20, for the choice τ_0 equal to 2.8. This diagram should be compared with Fig. 1*a*.

Notice, for $\tau \gg \tau_0$, that all solutions approach a form independent of τ_0 (and therefore α_0). In this power-law regime,

$$b(\mu, \tau) = (2\tau)^{-1/2} Z(\mu) , \tag{35}$$

and we cannot even properly speak of *any* intrinsic time scale.

V. DISCUSSION

In this paper, we have formulated and solved the problem of ambipolar diffusion in self-gravitating isothermal layers of lightly ionized gases. The most important result to emerge from our analysis is not so much the detailed finite-difference solutions, but the recognition of a pattern of asymptotic behavior for the magnetic field in time. We discuss here how general this result might be.

The separation of the spatial and temporal parts of equation (29) shows that the inverse square root decay in time of the magnetic field is the inevitable property of the asymptotic behavior whenever we have one-dimensional diffusion of ions and magnetic field in a *fixed* background of neutrals, independent of the details of the spatial structure of that background. This explains why in a very different problem, Mouschovias and Paleologou also obtained an inverse square root decay when they assumed instantaneous ionization balance. The crucial element needed to derive this law is that in equation (15), ρ and ρ_i have no explicit dependence on time. This is asymptotically true in any problem where the final state (with no magnetic field threading the neutral gas) is a stable equilibrium.

In a situation where ions are neither created nor destroyed, Mouschovias and Paleologou (1981) considered the case ρ = constant, but $\rho_i \propto B$. Here, they found the asymptotic dependence of B to be $1/t$, a result which is also easily derivable from equation (15). Indeed, the predictive power of equation (15) (whose validity does *not* depend on the assumption of magnetohydrostatic equilibrium for the neutrals) with respect to the issue of the asymptotic behavior of the field is so great that we are tempted to think that its generalization to two and three dimensions would be extremely helpful in answering the important question of how much interstellar magnetic flux is ultimately trapped in forming stars. However, that is a program which needs to be left to future investigations.

It is a pleasure to thank John Bahcall for his unflagging enthusiasm, welcome encouragement, and steady advice. Lyman Spitzer's comments added perspective to this paper. I am grateful to C. C. Lin for suggesting the terminology "shape-invariant" to describe asymptotic solutions for which "self-similar" and "homologous" would be misnomers. Over many conversations, Telemachos Mouschovias has educated me on the intricacies of interstellar magnetic fields. This paper was inspired by trying to understand his 1981 paper with Paleologou. The final spur to think seriously was supplied by Bruce Draine's asking me how much interstellar magnetic field might be incorporated in protostars; in addition, he provided several useful references. After a first draft of this paper was written, Howie Scott informed me that he was working independently on a similar problem and had obtained related results. Finally, I appreciate helpful discussions with Dave Gilden, Arieh Königl, Nick Kylafis, and, expecially, Wayne Roberge. This work was supported by the Institute for Advanced Study during a sabbatical leave from the University of California at Berkeley and by NSF grant PHY 79-19884.

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