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ON THE STATISTICAL MECHANICS OF VIOLENT RELAXATION

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ABSTRACT

We reexamine the foundations of Lynden-Bell's statistical mechanical discussion of violent relaxation in collisionless stellar systems. We argue that Lynden-Bell's formulation in terms of a continuum description introduces unnecessary complications, and we consider a more conventional formulation in terms of particles. We then find the exclusion principle discovered by Lynden-Bell to be quantitatively important only at phase densities where two-body encounters are no longer negligible. Since the dynamical basis for the exclusion principle vanishes in such cases anyway, Lynden-Bell statistics always reduces in practice to Maxwell-Boltzmann statistics when applied to stellar systems. Lynden-Bell also found the equilibrium distribution function generally to be a sum of Maxwellians with velocity dispersions dependent on the phase density at star formation. We show that this difficulty vanishes in the particulate description for an encounterless stellar system as long as stars of different masses are initially well mixed in phase space. Our methods also demonstrate the equivalence between Gibbs's formalism which uses the microcanonical ensemble and Boltzmann's formalism which uses a coarse-grained continuum description. In addition, we clarify the concept of irreversible behavior on a macroscopic scale for an encounterless stellar system. Finally, we comment on the use of unusual macroscopic constraints to simulate the effects of incomplete relaxation.

Subject heading: stars: stellar dynamics

I. INTRODUCTION

The stellar dynamical mechanism of violent relaxation has been invoked in an ever increasing number of contexts—among them, the formation and merger of galaxies (see the review by Gott 1977). Most studies, starting with Hénon's (1964) study of the relaxation process which occurs when the mean gravitational field of a spherical system changes violently, have been in the form of N-body calculations (see the review by Aarseth and Lecar 1975). Lynden-Bell's (1967) analytical investigation using the methods of statistical mechanics is a noteworthy exception. Despite the obvious shortcoming that any *unbounded* stellar system with a finite mass and a noncollapsed spatial structure cannot be completely relaxed, Lynden-Bell's discussion has provided coherent explanations for many aspects of the violent relaxation process, and for this reason it has deservedly become the classic paper in this field.

It is generally perceived that Lynden-Bell demonstrated the following points:

1. Encounterless relaxation via chaotic changes of the collective gravitational field, involving masses which are much greater than any individual stellar mass, leads to a final distribution whose properties are independent of stellar mass. Consequently, no mass segregation results from the operation of the violent relaxation process.

2. An exclusion principle results if two-body encounters are ignored, and the resulting equilibrium distribution formally resembles the Fermi-Dirac distribution. In the limit relevant to realistic stellar systems, the distribution becomes the Maxwell-Boltzmann distribution, but with stars of different masses having the same velocity dispersion and not the same "temperature."

3. The formal resemblance of Lynden-Bell statistics to Fermi-Dirac statistics is largely coincidental inasmuch as stars, being classical objects, are distinguishable particles. Moreover, the attractive interaction provided by the collective gravitational field makes a "gas" of stars a nonideal gas. It is especially provocative to think of the tendency of well relaxed stellar systems (either collisional or collisionless) to suffer a "gravothermal catastrophe" (see Spitzer 1974) as a phenomenon analogous to phase transition in the classical thermodynamic sense (see Lynden-Bell and Wood 1968).

Unfortunately, it has gone largely unnoticed that Lynden-Bell's actual analysis left two strings dangling. The first loose end concerns Lynden-Bell's derivation for the condition of the onset of degeneracy. Lynden-Bell's criterion involves the comparison of the final density in phase space with that applicable at star formation. Clearly, apart from the conserved label(s) of mass (charge, spin, etc.) of the particles, the statistical mechanics of any system of particles should not need to refer to the mechanism of the formation of those particles. The approach to equilibrium, i.e., the kinetic description of the transient stages, may depend on the initial conditions, but the final state can remember only those invariants which are expressed as explicit constraints on the system.

The second loose end concerns Lynden-Bell's conclusion in his Appendix I that the general case of his new statistics leads to "a superposition of Maxwellian components whose (squares of the) velocity dispersions are inversely proportional to the phase density of the component at star formation." Cuperman, Harten, and Lecar (1971) have claimed approximate verification of this result on the basis of some numerical experiments on collisionless one-dimensional stellar systems. Unfortunately, the number of distinct pieces of phase space studied by these authors is small, and their results seem equally consistent with a prediction of a uniform velocity dispersion. In any case, we show that Lynden-Bell's findings on these two points lack a clear dynamical basis.

We argue in this paper that Lynden-Bell was artificially forced into his difficulties because he chose not to recognize explicitly the particulate nature of stellar systems and developed his statistical arguments for a continuum. Lynden-Bell's (1967) paper was motivated (Lynden-Bell 1978, private communication) by the attempt to answer the following question: Among all fine-grained distribution functions with the same invariants, which coarse-grained distribution is the most common? In particular, his statistical mechanical method attempted to incorporate explicitly the constraint provided by the Liouville-Boltzmann equation for an encounterless stellar system (see eq. [57]): the fine-grained density of stars in 6-dimensional phase space (μ-space) is a conserved integral of the motion of a star (Jeans's theorem). To reduce the problem to one of counting states, Lynden-Bell introduced the concept of elementary volumes of μ-space with different densities and therefore different masses. Unfortunately, in our opinion, these elementary volumes play exactly the role of compound particles. In the next to simplest case of two different phase densities, it is as if Lynden-Bell had said that a galaxy is composed only of two kinds of star clusters: globular clusters and open clusters. In such a model, one never needs to refer to the truly elementary particles (the stars), which move together as a unit inside the compound particle (the cluster). Only the pieces of phase space of different densities and masses count as they (the clusters) interact with one another and with the violently changing collective gravitational field. If the relaxation mechanism is not specified *a priori* to be collisionless, equipartition of energy (Lynden-Bell's Appendix I) would naturally and correctly predict that the distribution of globular clusters would be more centrally concentrated than the distribution of open clusters. This conclusion would have no mechanistic basis, however, if the clusters are so many in number as to undergo only violent relaxation.

We believe that Lynden-Bell's preoccupation with Jeans's theorem is (largely) misplaced because this constraint can (mostly) be ignored for the statistics of a *macroscopic* description. A similar situation arises in classical statistical mechanics where Liouville's theorem for an ensemble of isolated systems (see eq. [3]) constrains the density of *system* points in $6N$-dimensional phase space (Γ-space) to be a conserved integral of the motion of a system point. In that case, Gibbs and Boltzmann argued convincingly that, independent of the initial distribution of system points in Γ-space, almost all ensembles manage, *upon coarse-graining*, to look more and more like the microcanonical ensemble (uniform Γ-space density) because of the phase mixing of system points in Γ-space. Because the stars within our stellar system are interacting with the collective gravitational field if not with each other, Jeans's theorem has a somewhat different interpretation than Liouville's theorem, but the principle of ignorable integrals of motion should still apply.

Our approach, therefore, is to adopt the particulate description from the outset and refer explicitly to a system of N stars. The formal mathematical problem that we treat differs from Lynden-Bell's problem, but we refer, of course, to the same astronomical systems. We then find that a description in terms of the distribution of an ensemble of systems in Γ-space and the distribution of particles in μ-space along the lines of Gibbs and Boltzmann's ideas (cf. Uhlenbeck and Ford 1963) serves to tie up both loose ends of Lynden-Bell's arguments with a single knot. Our analysis also makes explicit the conditions under which conclusions (1) and (2) are valid and under which the much simpler methods of Boltzmann involving only the coarse-grained continuum phase density can be applied. To avoid too many cross references, we have made the discussion of the present paper self-contained.

II. THE MICROCANONICAL ENSEMBLE

An accurate description of an N-body stellar system follows the positions and momenta of every particle (e.g., a numerical N-body calculation). For an isolated system (not necessarily a single galaxy, the system could contain all the stars of two interacting galaxies, or even of a cluster of galaxies), the dynamical behavior is prescribed by a Hamiltonian which, written in Cartesian coordinates and conjugate momenta, reads

$$H_s = \sum_{i=1}^{N} \frac{|\boldsymbol{p}_i|^2}{2m_i} - \frac{1}{2} \sum_{i,k=1,i\neq k}^{N} \frac{Gm_i m_k}{|\boldsymbol{x}_i - \boldsymbol{x}_k|} . \tag{1}$$

Since the system Hamiltonian does not depend explicitly on time, the total energy of the system is conserved:

$$H_s(\{\boldsymbol{x}_i\}, \{\boldsymbol{p}_i\}) = E . \tag{2}$$

We henceforth work in the frame of the center of mass so that E does not contain the kinetic energy associated with a uniform translation of the system.

Let Γ-space be the $6N$-dimensional phase space whose mutually orthogonal axes label the $3N$ spatial coordinates and the $3N$ momenta coordinates of the N particles. The exact state of an N-body system is given by a single point in Γ-space. We now follow Gibbs and consider an ensemble of such isolated N-body systems, all with the same energy E. It is well known that the density of system points in Γ-space, $D_e(\{x_i\}, \{p_i\}, t)$, evolves in time in accordance with Liouville's theorem,

$$\frac{\partial D_e}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial H_s}{\partial \boldsymbol{p}_i} \cdot \frac{\partial D_e}{\partial \boldsymbol{x}_i} - \frac{\partial H_s}{\partial \boldsymbol{x}_i} \cdot \frac{\partial D_e}{\partial \boldsymbol{p}_i}\right) = 0 . \tag{3}$$

The basic postulate of classical statistical mechanics is that, given coarse enough resolution and the passage of sufficient time, any initial collection of system points will spread, because of interactions within each system, to cover uniformly the energy hypersurface in Γ-space defined by equation (2). The interactions need not be restricted to particle-particle encounters; they may well involve the response of particles to a violently and randomly changing collective gravitational field. In any case, the ensemble of isolated systems having the property $D_e =$ constant on the energy hypersurface (2), is called the microcanonical ensemble. A system selected at random from the microcanonical ensemble will have equal probability of being at any system point consistent with the constraints.

The applicability of the microcanonical ensemble to our problem is worrisome because N-body experiments show that violently changing gravitational fields of finite duration and limited spatial extent do not suffice to make every possible microstate equally probable. Some of the necessary modifications can be introduced by including macroscopic constraints additional to equation (2). In §§ III, IV, and V, we adopt the microcanonical ensemble generalized in this manner as the fundamental basis for making probabilistic arguments.

Ensemble averages are convenient for the calculation of the properties of equilibrium systems and their fluctuations (see Huang 1963). To study the macroscopic properties of a given system, equilibrium or non-equilibrium, however, it is more useful to introduce a smoothed-out mass density of particles in 6-dimensional μ-space:

$$F(\boldsymbol{x}, \boldsymbol{v}, t)d^3xd^3v = \text{mass of stars at time } t \text{ within } d^3xd^3v \text{ centered on } (\boldsymbol{x}, \boldsymbol{v}) . \tag{4}$$

To obtain a smooth function F from a distribution of N discrete particles, we necessarily have to lump large numbers of stars into groups of similar masses, positions, and velocities. We start in the next section with the simplest case where all N stars have the same mass m, and we describe the practical groupings into positions and velocities. We then relate the Γ-space and the μ-space descriptions.

III. STATISTICAL MECHANICS OF N STARS OF EQUAL MASS

a) The Microscopic Description and Exclusion Principle

We first divide μ-space into an enormous number of fixed microcells of 6-dimensional volume g each. The microcell volume g is chosen small enough that each microcell has at most one star in it. (It may also have none; we ignore the problem of binary stars.) We also pick g large enough that two stars at adjacent microcells will not suffer a two-body encounter which would make them travel, in a dynamical crossing time (the time scale associated with violent relaxation, Lynden-Bell 1967) to a different microcell from the one into which they would have gone in the absence of such an encounter. For stellar systems with $N \gg 10^2$ (cf. Spitzer 1974), such microcells can be defined with g being very small in comparison with the totality of the volume of μ-space available to the system. For example, in the Galaxy, we could use a grid of microcells with spatial and velocity intervals of a fraction of a parsec and a fraction of a kilometer per second.

When the number N of point stars in a system is very large, the dynamics of each star is governed by a particle Hamiltonian per unit mass which depends only on the position and velocity of the star in question and not explicitly on the μ-space location of any other single star. Since the motion is then completely deterministic, two stars which initially occupy different microcells will continue to occupy different microcells. Once we have set up our fine mesh of microcells properly, we are guaranteed that each microcell is occupied by at most one star for the duration of the period of violent relaxation. This microscopic exclusion principle explains why our choice of the microcell volume g differs from Lynden-Bell's definition ϖ which refers to star formation.

b) The Macroscopic Description

Now group together a large number ν_1 of neighboring microcells in one corner of μ-space to form a macrocell of volume $\omega_1 = \nu_1 g$. The number ν_1 is chosen large enough that the number $n_1 \leq \nu_1$ of occupied microcells (i.e., the number of stars) in the macrocell is large enough to do meaningful statistics (see eq. [25] and the comments which follow it). However, n_1 must still be small enough, in comparison with the total number of stars N, that the collective gravitational potential is mostly due to the stars outside the macrocell (see eq. [6]). Continue to group large numbers of microcells together in this way until at the opposite corner of μ-space we form the Mth macrocell of volume $\omega_M = \nu_M g$. The set $\{\omega_a\}$ constitutes our coarse mesh of μ-space, and the set of occupation numbers $\{n_a\}$ defines a macrostate.

c) Correspondence between Γ-Space and μ-Space Descriptions

Since we deal with isolated systems (or with systems confined inside reflecting walls), the set of occupation numbers $\{n_a\}$ satisfy the macroscopic constraints

$$\sum_{a=1}^{M} mn_a = Nm\,, \tag{5a}$$

$$\sum_{a=1}^{M} mn_a(\tfrac{1}{2}|\boldsymbol{v}_a|^2 + \tfrac{1}{2}\Phi_a) = E \pm \Delta E\,, \tag{5b}$$

where

$$\Phi_a \equiv \Phi(\boldsymbol{x}_a, t) \equiv -\sum_{b=1, b\neq a}^{M} \frac{Gmn_b}{|\boldsymbol{x}_a - \boldsymbol{x}_b|} \tag{6}$$

is the macroscopic gravitational field, and $(\boldsymbol{x}_a, \boldsymbol{v}_a)$ is the center of the ath macrocell. A factor of $\frac{1}{2}$ in front of Φ_a is introduced in equation (5b) to offset counting pairs of macrocells twice, and the total energy has a slight variation $\pm\Delta E$ about E because the finite size of ω_a introduces into the ensemble, systems with macrostates whose energies are not exactly E. This uncertainty is unimportant for the formal development of the theory, and we ignore it in what follows except we shall speak of an energy shell in Γ-space instead of an energy (hyper)surface.

We now note that each system point in Γ-space corresponds to a unique macrostate in μ-space, but the reverse is not true. In fact, a given macrostate $\{n_a\}$ in μ-space corresponds to a whole region in Γ-space with the $6N$-dimensional volume:

$$W(\{n_a\}) = \left[\frac{N!}{n_1!\cdots n_M!}\right]\left[\frac{\nu_1!\cdots\nu_M!}{(\nu_1 - n_1)!\cdots(\nu_M - n_M)!}\right][m^{3N}g^N]\,. \tag{7}$$

Our proof of equation (7) can be broken down to three steps:

A) Notice that the μ-space distribution does not change if we move a star around in its microcell; however, such jiggling generates a subvolume m^3g in Γ-space. (The factor m^3 enters because we have defined μ-space in terms of positions and velocities and Γ-space in terms of positions and momenta.) Moving another star around in its microcell generates another subvolume m^3g in orthogonal directions to the first. Moving all N stars around simultaneously in their microcells generates the volume

$$m^{3N}g^N\,. \tag{8}$$

B) Notice also that the μ-space distribution does not change if we permute the n_a stars in the ath macrocell among the ν_a microcells. Since the stars are distinguishable particles, the number of ways of putting n_a stars into ν_a bins is $\nu_a!/(\nu_a - n_a)!$. Hence, without interchanging any stars from macrocell to macrocell, the number of ways of permuting stars into the available microcells is

$$\frac{\nu_1!\cdots\nu_M!}{(\nu_1 - n_1)!\cdots(\nu_M - n_M)!}\,. \tag{9}$$

C) Finally, notice that the μ-space distribution does not change if we interchange stars in different macrocells keeping the set of occupation numbers $\{n_a\}$ fixed. The number of ways to divide N distinguishable stars into the set $\{n_a\}$ is

$$\frac{N!}{n_1!\cdots n_M!}\,. \tag{10}$$

Carrying out all possible combinations of the independent operations A, B, C, we obtain the total volume occupied by the macrostate $\{n_a\}$ as equation (7). (Q.E.D.)

As an aside, we note that the *microstate* discussed in § III*a* occupies a volume

$$w = N!m^{3N}g^N \tag{11}$$

in Γ-space, where the factor $N!$ comes from the fact that we have N distinguishable particles. The expression (11) is much less than equation (7) unless $n_a = \nu_a$ for all a, i.e., unless all available microstates are filled. In the non-degenerate limit, $n_a \ll \nu_a$, many microstates correspond to the same macrostate. The volume (11) occupied by each microstate is fixed, whereas the volume (7) occupied by a macrostate will change greatly if the occupation numbers n_a are varied. This fact is related to the concept of irreversible behavior on the macroscopic level despite the reversibility of the equations of motion on a microscopic level (see § V). Translated into probabilistic arguments for a single system, we may say that a given system spends much more time in some macrostates than in others.

We also note that the "third law of thermodynamics" is satisfied by defining the macroscopic entropy S to be $k \ln (W/w)$ instead of the usual $k \ln W$. Fixing the completely degenerate state as the zero-point of the entropy is, however, inconsequential for classical statistical mechanics, and we ignore this convention in what follows.

d) The Lynden-Bell Distribution

Let us now calculate the macrostate $\{n_a\}$ which occupies the largest piece $W(\{n_a\})$ of the energy shell in Γ-space, consistent with the constraints (5). Since the logarithm is a monotonic function, we can perform this calculation by varying the occupation numbers n_a to maximize the expression

$$\ln [W(\{n_a\})] - \alpha \sum_{a=1}^{M} mn_a - \beta \sum_{a=1}^{M} mn_a(\tfrac{1}{2}|\boldsymbol{v}_a|^2 + \tfrac{1}{2}\Phi_a) , \tag{12}$$

where α and β are Lagrange multipliers introduced to remove the constraints on the independence of the variations of the n_a's. Substituting equation (7), using Stirling's approximation $\ln (n!) \approx n(\ln n - 1)$ for $n \gg 1$, and taking the variation of equation (12) yield

$$\sum_{a=1}^{M} [-\ln n_a + \ln (\nu_a - n_a) - \alpha m]\delta n_a - \beta m \sum_{a=1}^{M} [(\tfrac{1}{2}|\boldsymbol{v}_a|^2 + \tfrac{1}{2}\Phi_a)\delta n_a + \tfrac{1}{2}n_a\delta\Phi_a] = 0 . \tag{13}$$

Now, equation (6) implies that

$$\delta\Phi_a = - \sum_{b=1, b\neq a}^{M} \frac{Gm\delta n_b}{|\boldsymbol{x}_a - \boldsymbol{x}_b|} ; \tag{14}$$

from which we easily obtain the identity

$$\sum_{a=1}^{M} mn_a\delta\Phi_a = \sum_{a=1}^{M} m\Phi_a\delta n_a . \tag{15}$$

Substitution of equation (15) into equation (13) now gives the requirement

$$\ln [(\nu_a - n_a)/n_a] = \alpha m + \beta m\epsilon_a , \tag{16}$$

where ϵ_a is the energy per unit mass of a star in the center of the ath macrocell:

$$\epsilon_a = \tfrac{1}{2}|\boldsymbol{v}_a|^2 + \Phi_a . \tag{17}$$

Define the chemical potential per unit mass, $\mu \equiv -\alpha/\beta$ (not to be confused with the notation for 6-dimensional phase space), and write the distribution (16) as

$$n_a = \frac{\nu_a}{1 + \exp [\beta m(\epsilon_a - \mu)]} , \tag{18}$$

which, apart from the interpretation for the microcell volume, is the Fermi-Dirac distribution. We call the expression (18), which maximizes $W(\{n_a\})$, the Lynden-Bell distribution.

e) Lack of Degeneracy

Now "degeneracy effects" are important if $\exp [\beta m(\epsilon_a - \mu)]$ is on the order of unity. From equation (18) we see that this requires $n_a \approx \nu_a$, i.e., for all the available microcells in a macrocell to be nearly occupied. Given our definition of g, this can happen only if two-body encounters are important. On the one hand, if two-body encounters *are* important, we do not have an exclusion principle, and Maxwell-Boltzmann statistics should have been used instead of Lynden-Bell statistics. On the other hand, if two-body encounters are *not* important, then $n_a \ll \nu_a$, and equation (18) requires $\exp [\beta m(\epsilon_a - \mu)] \gg 1$, i.e.,

$$n_a \approx \nu_a \exp [-\beta m(\epsilon_a - \mu)] , \tag{19}$$

which is the Maxwell-Boltzmann distribution. Thus, in all practical applications to stellar systems, we need only concern ourselves with Maxwell-Boltzmann statistics. The important difference between collisionless and collisional relaxation is the issue of whether or not the "temperature" is proportional to the stellar mass. We address this point in § IV when we consider more than one mass species. We defer until § V the discussion of the difficulties of self-gravitating "isothermal" distributions.

In any case, for $n_a \ll \nu_a$, we can approximate $\nu_a!/(\nu_a - n_a)!$ as $\nu_a{}^{n_a}$ in equation (7). With $\omega_a \equiv \nu_a g$, we can now write the volume in Γ-space occupied by the macrostate $\{n_a\}$ as

$$W(\{n_a\}) = \frac{N!m^{3N}}{n_1!\cdots n_M!}\,\omega_1{}^{n_1}\cdots\omega_M{}^{n_M}\,, \tag{20}$$

an expression which makes no reference to microcells of size g.

f) Statistical Interpretation and Fluctuations

As long as we speak of $W(\{n_a\})$ in terms of a volume in Γ-space, everything that we have done so far is rigorously correct. It is the identification of $W(\{n_a\})$ as a measure of probability that contains the statistical mechanical assumption that the Γ-space energy shell is uniformly populated by system points. *If* the microcanonical ensemble is a legitimate basis for the statistics, *then* a system selected at random from the ensemble is most likely to be in the macrostate corresponding to the "isothermal" distribution (19), which occupies the most volume in Γ-space. Moreover, the statistical fluctuations from this "equilibrium" macrostate will be small because the bulk of the volume of the energy shell turns out to be occupied by the equilibrium macrostate.

To see this, let us consider the mean-square deviation of the occupation number n_b:

$$\langle (n_b - \langle n_b\rangle)^2\rangle = \langle n_b{}^2\rangle - \langle n_b\rangle^2\,, \tag{21}$$

where angular brackets denote ensemble averages:

$$\langle n_b\rangle = \frac{1}{\Gamma}\sum_{\{n_a\}} n_b W(\{n_a\})\,, \tag{22a}$$

$$\langle n_b{}^2\rangle = \frac{1}{\Gamma}\sum_{\{n_a\}} n_b{}^2 W(\{n_a\})\,, \tag{22b}$$

$$\Gamma \equiv \sum_{\{n_a\}} W(\{n_a\})\,, \tag{22c}$$

with the sums in equations (22) extending over all possible sets of the occupation numbers $\{n_a\}^M{}_{a=1}$ compatible with the constraints (5). Comparing equations (20) and (22), we easily obtain

$$\langle n_b\rangle = \frac{\omega_b}{\Gamma}\frac{\partial\Gamma}{\partial\omega_b}\,, \tag{23a}$$

$$\langle n_b{}^2\rangle = \frac{\omega_b}{\Gamma}\frac{\partial}{\partial\omega_b}\left(\omega_b\frac{\partial\Gamma}{\partial\omega_b}\right) = \frac{\omega_b}{\Gamma}\frac{\partial}{\partial\omega_b}(\Gamma\langle n_b\rangle)\,. \tag{23b}$$

For the equilibrium distribution (19), n_a is proportional to ω_a. Let us assume that $\langle n_b\rangle$ is likewise proportional to ω_b. Then equation (23b) gives

$$\langle n_b{}^2\rangle = \frac{\omega_b}{\Gamma}\frac{\partial\Gamma}{\partial\omega_b}\langle n_b\rangle + \omega_b\frac{\partial\langle n_b\rangle}{\partial\omega_b} = \langle n_b\rangle^2 + \langle n_b\rangle\,, \tag{24}$$

which together with equation (21) implies that the fractional rms deviation,

$$\frac{\langle (n_b - \langle n_b\rangle)^2\rangle^{1/2}}{\langle n_b\rangle} = \langle n_b\rangle^{-1/2}\,, \tag{25}$$

will be small if $\langle n_b\rangle \gg 1$. Given that the latter is true, most of the systems in the microcanonical ensemble must be virtually indistinguishable from those that correspond to the most probable macrostate; i.e., the equilibrium macrostate (19) must occupy almost the entire volume of the Γ-space energy shell. Our assumption that $\langle n_b\rangle$ is proportional to ω_b is therefore justified *a posteriori*. (Q.E.D.)

The formula (25) suggests that a viable macroscopic description necessarily involves a substantial loss of resolution for realistic stellar systems. Even for a giant spherical galaxy of 10^{12} stars, if we partition 6-dimensional phase space into 32 segments on a side, we will have about 10^3 stars in a typical macrocell. In other words, if we want to keep statistical fluctuations down to a level of a few percent in a typical macrocell, we need to adopt the relatively coarse spatial and velocity resolutions of several hundred parsecs and a few tens of kilometers per second.

As an aside, we note that statistical fluctuations are crudely taken into account in conventional estimates for the "two-body" relaxation time scale by setting the maximum impact parameter equal to the typical linear extent of the entire stellar system (see, e.g., Woltjer 1967). The corrections are never very large, and our discussion in

§ III*a* of the role of two-body encounters is insensitive to the precise refinements used to estimate the encounter relaxation time scale.

IV. STATISTICAL MECHANICS OF N STARS WITH J MASSES

In § III we confined the discussion to N stars of a single mass m. Consider now an encounterless stellar system with $N^{(j)}$ stars of masses $m^{(j)}$ such that

$$\sum_{j=1}^{J} N^{(j)} = N\,, \qquad \text{and} \qquad \sum_{j=1}^{J} N^{(j)} m^{(j)} = Nm\,, \tag{26}$$

where N is the total number of stars and m is the average mass of a star in the system. We assume $J \ll N$. We may again divide μ-space into M macrocells of volumes $\omega_1, \ldots, \omega_M$. We now specify the macrostate by the J sets of occupation numbers

$$\{n_a^{(j)}\}^M{}_{a=1} \qquad \text{for } j = 1, \ldots, J\,, \tag{27}$$

which are subject to the constraints

$$\sum_{a=1}^{M} n_a^{(j)} = N^{(j)}\,, \tag{28a}$$

$$\sum_{j=1}^{J} \sum_{a=1}^{M} m^{(j)} n_a^{(j)} (\tfrac{1}{2}|\boldsymbol{v}_a|^2 + \tfrac{1}{2}\Phi_a) = E \pm \Delta E\,, \tag{28b}$$

with Φ_a given by

$$\Phi_a = -\sum_{j=1}^{J} \sum_{b=1, b\neq a}^{M} \frac{G m^{(j)} n_b^{(j)}}{|\boldsymbol{x}_a - \boldsymbol{x}_b|}\,. \tag{29}$$

Ignoring the exclusion principle for the reasons given in § III*e*, we find that the Γ-space volume occupied by this macrostate is (cf. eq. [20])

$$W = \prod_{j=1}^{J} N^{(j)}! [m^j]^{3N^{(j)}} \prod_{a=1}^{M} [\omega_a^{n_a^{(j)}} / n_a^{(j)}!]\,. \tag{30}$$

Now, if equations (28) were the only constraints that our system had to satisfy, we would find that the most probable distribution is a sum of J true Maxwellians with the same temperature for each mass distribution, i.e., with the squares of the velocity dispersion being inversely proportional to the mass $m^{(j)}$. This result is formally related to Lynden-Bell's (1967) difficulties in Appendix 1 of his paper (see § I). It obtains because no explicit recognition has yet been given that the system is collisionless and is likely to have a mass function which is well mixed in phase space. For a collisionless stellar system, stars of all masses have nearly identical motions if they have nearly identical μ-space locations. The simplest assumption we can make is that the mass function was initially well mixed. The equations of motion then guarantee the mass function to continue in time to have a uniform form throughout μ-space:

$$n_a^{(j)} = \frac{N^{(j)}}{N} n_a \qquad \text{for } j = 1, \ldots, J\,, \tag{31}$$

where n_a is the total number of stars in the ath macrocell:

$$n_a = \sum_{j=1}^{J} n_a^{(j)}\,. \tag{32}$$

With the constraints (31) and the formula (30), we may derive the entropy S associated with the macrostate (27) as

$$S \equiv k \ln(W) = -k \sum_{j=1}^{J} \sum_{a=1}^{M} n_a^{(j)} \ln[n_a^{(j)}/\omega_a] + \text{const.} = -k \sum_{a=1}^{M} n_a \ln(n_a/\omega_a) + \text{const.}\,, \tag{33}$$

where k is Boltzmann's constant. The additive constants in the above expressions do not depend on the variables n_a and are unimportant for the definition of the entropy in classical statistical mechanics. (We ignore the question of "correct Boltzmann counting.") We now define the coarse-grained distribution function $F(\boldsymbol{x}, \boldsymbol{v}, t)$ via (cf. eq. [4]):

$$F(\boldsymbol{x}_a, \boldsymbol{v}_a, t)\omega_a = \sum_{j=1}^{J} m^{(j)} n_a^{(j)} = m n_a\,, \tag{34}$$

where m is the average mass defined by equation (26) and $(\boldsymbol{x}_a, \boldsymbol{v}_a)$ is the center of the ath macrocell. Substituting equation (34) into equation (33), approximating the sum by an integral, and dropping an inconsequential additive constant, we write the coarse-grained entropy as

$$S = -\frac{k}{m}\int F \ln F d^3x d^3v\,, \tag{35}$$

which is, of course, Boltzmann's definition. Maximizing W subject to the constraints (28) and (31) is now equivalent to maximizing S subject to the constraints

$$\int F d^3x d^3v = \text{constant}\,, \tag{36a}$$

$$\int (\tfrac{1}{2}|v|^2 + \tfrac{1}{2}\Phi) F d^3x d^3v = \text{constant}\,. \tag{36b}$$

The latter mathematical task is easy and results in the single Maxwellian,

$$F = A \exp(-\epsilon/c_0^2)\,, \tag{37}$$

where A and c_0 are constants and ϵ is the energy per unit mass of a star:

$$\epsilon = \tfrac{1}{2}|v|^2 + \Phi\,. \tag{38}$$

Notice that the assumption, equation (31), that the mass function is well mixed in μ-space reduces the encounterless problem with J mass species to an equivalent N-body problem with a single mass m. In particular, the equilibrium distribution function (37) is a single Maxwellian with a uniform velocity dispersion c_0—as we would have expected intuitively. For the distribution (37), *violent* relaxation has gone to completion, but the constraints (31) keep the system from being completely relaxed in a true *thermodynamic* sense (uniform temperature). Notice also that we have cavalierly passed to a continuum description. The route of this passage is important for the concept of irreversible behavior on the macroscopic level and is examined more carefully in the next section. The basic issue that we address in § V is how to derive macroscopic equations which contain an "arrow of time" from microscopic equations which are intrinsically time reversible.

V. THE APPROACH TO EQUILIBRIUM

It is well known that the entropy of a classical gas increases in time only because of the unpredictability of the action of molecular collisions on a macroscopic level. It might therefore be thought that the entropy of a collisionless stellar system must remain constant and cannot be maximized to discover the final state (even if violent relaxation could go to completion). We answer this objection below.

a) Fine-grained, Coarse-grained, and Continuum Distributions

To discuss the evolution in time of a given system toward equilibrium, which is the province of kinetic theory, it is standard practice to use differentiable single-particle distribution functions. For an encounterless stellar system, there are a number of different kinds of functions which one can introduce. For simplicity, we discuss this issue in the context of N particles of mass m since generalization along the lines of the previous section is trivial.

Mathematically, the most satisfactory approach is to use the BBGKY hierarchy of kinetic equations (see Uhlenbeck and Ford 1963 or Montgomery and Tidman 1964). Consider an ensemble of systems with a distribution of microstates to be specified later. Our ensemble is, at least initially, not necessarily the microcanonical ensemble (cf. § II), and the Γ-space phase density, D_e, of our ensemble is generally nonuniform on the energy hypersurface (2). The probability $P_N d^3x_1 d^3v_1 \cdots d^3x_N d^3v_N$ of finding a system chosen at random from the ensemble at time t with its N particles within volume $d^3x_1 d^3v_1 \cdots d^3x_N d^3v_N$ centered about $(\boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{v}_N)$ is given via

$$P_N(\boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{v}_N, t) = \frac{m^{3N} D_e(\{\boldsymbol{x}_i\}, \{\boldsymbol{p}_i\}, t)}{\int D_e d^3x_1 d^3p_1 \cdots d^3x_N d^3p_N}\,, \tag{39}$$

where $m\boldsymbol{v}_i = \boldsymbol{p}_i$.

The denominator of equation (39) corresponds to the total number of systems in our ensemble and is fixed. Therefore, Liouville's theorem (3) can also be written as

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N}\left[\boldsymbol{v}_i\cdot\frac{\partial P_N}{\partial \boldsymbol{x}_i} - \frac{\partial V}{\partial \boldsymbol{x}_i}\cdot\frac{\partial P_N}{\partial \boldsymbol{v}_i}\right] = 0\,, \tag{40}$$

where

$$-\frac{\partial V}{\partial \boldsymbol{x}_i}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \equiv \frac{\partial}{\partial \boldsymbol{x}_i}\left(\sum_{k=1, k\neq i}^{N} \frac{Gm}{|\boldsymbol{x}_i - \boldsymbol{x}_k|}\right)\,. \tag{41}$$

Now let $P_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t)$ be the probability density at time t associated with finding a single star (star 1) at $(\boldsymbol{x}_1, \boldsymbol{v}_1)$ independent of the locations of the other stars in the system. Clearly,

$$P_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t) = \int P_N d^3x_2 d^3v_2 \ldots d^3x_N d^3v_N \,. \tag{42}$$

Integrate equation (40) over $(\boldsymbol{x}_2, \boldsymbol{v}_2, \ldots, \boldsymbol{x}_N, \boldsymbol{v}_N)$; assume P_N vanishes at the boundaries of phase space; and obtain

$$\frac{\partial P_1}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial P_1}{\partial \boldsymbol{x}_1} - \int \frac{\partial V}{\partial \boldsymbol{x}_1} \cdot \frac{\partial P_N}{\partial \boldsymbol{v}_1} d^3x_2 d^3v_2 \cdots d^3x_N d^3v_N = 0 \,. \tag{43}$$

Some sort of closure condition is needed if the hierarchy of kinetic equations, of which equation (43) is merely the first member, is to form a complete set at an order less than N. For an encounterless stellar system, the correct closure relation is the simplest possible, namely, that the N-particle probability density, P_N, is the product of N single-particle probability densities:

$$P_N(\boldsymbol{x}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{x}_N, \boldsymbol{v}_N, t) = P_1(\boldsymbol{x}_1, \boldsymbol{v}_1, t) \cdots P_1(\boldsymbol{x}_N, \boldsymbol{v}_N, t) \,, \tag{44}$$

where the normalization condition (39) requires

$$\int P_1(\boldsymbol{x}', \boldsymbol{v}', t) d^3x' d^3v' = 1 \,. \tag{45}$$

With the ansatz (44), equation (43) can be rewritten as

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} - \frac{\partial \phi}{\partial \boldsymbol{x}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 \,, \tag{46}$$

where we have dropped the subscript 1, and where we have defined the fine-grained *probabilistic* distribution function, f, and its associated potential, ϕ, by

$$f(\boldsymbol{x}, \boldsymbol{v}, t) \equiv NmP_1(\boldsymbol{x}, \boldsymbol{v}, t) \,, \tag{47a}$$

$$\phi(\boldsymbol{x}, t) \equiv -\frac{(N-1)}{N} \int \frac{Gf(\boldsymbol{x}', \boldsymbol{v}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x' d^3v' \,. \tag{47b}$$

With the evolution of f prescribed by equation (46), it is easily verified that the entropy of the fine-grained distribution,

$$s \equiv -\frac{k}{m} \int f \ln f d^3x d^3v \,, \tag{48}$$

remains fixed in time, i.e., that $ds/dt = 0$.

Introduce a coarse-grained distribution F by coarse-mesh averaging f,

$$F_a \equiv F(\boldsymbol{x}_a, \boldsymbol{v}_a, t) = \frac{1}{\omega_a} \int_{\omega_a} f(\boldsymbol{x}, \boldsymbol{v}, t) d^3x d^3v \,, \tag{49}$$

where $(\boldsymbol{x}_a, \boldsymbol{v}_a)$ is the center of the ath macrocell of μ-space volume ω_a. The gravitational potential Φ, and the macroscopic entropy S, associated with F are given by (cf. eqs. [6] and [35])

$$\Phi_a \equiv \Phi(\boldsymbol{x}_a, t) = -\sum_{b=1, b \neq a}^{M} \frac{G\omega_b F_b}{|\boldsymbol{x}_a - \boldsymbol{x}_b|} \,, \tag{50a}$$

$$S \equiv -\frac{k}{m} \sum_{a=1}^{M} \omega_a F_a \ln F_a \,. \tag{50b}$$

Now, consider a *given* stellar system whose macrostate is specified at time t by the M occupation numbers, $\{n_a \equiv F_a \omega_a / m\}$, on the mesh $\{\omega_a\}$. If we lack microscopic information about our system, the fairest method of estimating the *probable* future evolution of its *macrostate* is to construct an ensemble of systems at time t in which the n_a stars have a uniform probability of being found anywhere inside the macrocell ω_a. Thus, at time t, our ensemble has a fine-grained probability distribution f which is identical to the coarse-grained distribution F of our given system:

$$f(\boldsymbol{x}, \boldsymbol{v}, t) = F(\boldsymbol{x}_a, \boldsymbol{v}_a, t) \qquad \text{for } (\boldsymbol{x}, \boldsymbol{v}) \text{ inside } \omega_a \,. \tag{51}$$

Because the sizes ω_a of the macrocells are finite, stars within a macrocell do not move together in μ-space. This is especially true when the fine-grained potential ϕ changes violently in time and in space, but divergent motions ("phase mixing") occur even when ϕ is temporally steady and spatially smooth (Lynden-Bell 1967, see also § V*b*). We now show that violent relaxation and phase mixing both lead to an *increase* of the macroscopic entropy S (see also Tolman 1938, whose analogous argument was brought to my attention by Frank Valdes 1977, private communication).

After a time Δt, the differential motions of stars in each macrocell ω_a will cause f to differ slightly from F, say, by δf:

$$f(\boldsymbol{x}, \boldsymbol{v}, t + \Delta t) = F(\boldsymbol{x}_a, \boldsymbol{v}_a, t + \Delta t) + \delta f(\boldsymbol{x}, \boldsymbol{v}, t + \Delta t) \qquad \text{for } (\boldsymbol{x}, \boldsymbol{v}) \text{ inside } \omega_a\,. \tag{52}$$

Since F at time $t + \Delta t$ is given by a coarse-mesh average of f (cf. eq. [49]), we require

$$\int_{\omega_a} \delta f d^3x d^3v = 0\,. \tag{53}$$

Now, $S(t) = s(t)$ by construction, whereas $s(t + \Delta t) = s(t)$ because $ds/dt = 0$. Thus, the change in the macroscopic entropy is given by

$$\Delta S \equiv S(t + \Delta t) - S(t) = S(t + \Delta t) - s(t) = S(t + \Delta t) - s(t + \Delta t)\,. \tag{54}$$

We write equation (48) at time $t + \Delta t$ as

$$s(t + \Delta t) = -\frac{k}{m} \sum_{a=1}^{M} \int_{\omega_a} f(\boldsymbol{x}, \boldsymbol{v}, t + \Delta t) \ln [f(\boldsymbol{x}, \boldsymbol{v}, t + \Delta t)] d^3x d^3v\,. \tag{55}$$

Substituting equation (52), expanding to second order for small δf, and using equations (50b) and (53), we obtain

$$S(t + \Delta t) - s(t + \Delta t) = \frac{k}{m} \sum_{a=1}^{M} \frac{1}{2F_a} \int_{\omega_a} (\delta f)^2 d^3x d^3v\,. \tag{56}$$

Since the right-hand side is positive-definite and Δt is positive but otherwise arbitrary, equation (54) requires $dS/dt \geq 0$. (Q.E.D.) Notice that our "derivation" of the second law of thermodynamics assumes that microscopic information is absent at the *earlier* instant (see also Layzer 1967). Thus, our introduction of an "arrow of time" requires that we pose a physically plausible initial-value problem—not its time reverse.

During periods of violent relaxation, the fluctuating potential generates random rms differential velocities of order $|\Delta \boldsymbol{v}| \approx (\Delta t \partial\phi/\partial t)^{1/2} \approx (\Delta t |\boldsymbol{v}|^2/\tau)^{1/2}$, where $|\boldsymbol{v}|$ and τ are a typical speed and crossing time. If Δt is a small but finite fraction of τ, the ratio $\delta f/F_a$ is of order $|\Delta \boldsymbol{v}|/|\boldsymbol{v}| \approx (\Delta t/\tau)^{1/2}$, and the rate of change $\Delta S/\Delta t$ is of order Nk/τ. An analogy with Brownian motion in velocity space would yield "diffusion" and "dynamical friction" coefficients of order $|\boldsymbol{v}|^2/\tau$ and $1/\tau$; however, such an analogy is dubious here because all changes occur on the same time scale τ.

We should remark in passing that f in most discussions of encounterless stellar systems refers to the fine-grained distribution function of a single system and is therefore a sum of Dirac delta functions rather than the differentiable function of our definition. For many applications to single systems, the graininess of the actual distribution is unimportant, and it is convenient to work in a continuum approximation where we take the limits $N \to \infty$ and $m \to 0$, keeping the product Nm fixed. In this limit, the two-body relaxation time scale becomes infinite if the mass density remains finite, and we can fill up all of μ-space with a continuum of stars. Denote the continuum phase density obtained for f in this limit by ψ. The dynamical evolution of ψ is evidently governed by the Liouville-Boltzmann equation

$$\frac{\partial \psi}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \psi}{\partial \boldsymbol{x}} - \frac{\partial \phi}{\partial \boldsymbol{x}} \cdot \frac{\partial \psi}{\partial \boldsymbol{v}} = 0\,, \tag{57}$$

where ϕ is the associated self-consistent potential (cf. eq. [47b])

$$\phi(\boldsymbol{x}, t) = -\int \frac{G\psi(\boldsymbol{x}', \boldsymbol{v}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x' d^3v'\,. \tag{58}$$

Can we usefully introduce a function $\Psi(\boldsymbol{x}, \boldsymbol{v}, t)$ which is the continuum limit of $F(\boldsymbol{x}_a, \boldsymbol{v}_a, t)$? Yes, but only if the gravitational potential $\phi(\boldsymbol{x}, t)$ can be considered to be changing violently and randomly when the volume ω_a of each macrocell in the coarse mesh $\{\omega_a\}^M{}_{a=1}$ decreases to zero in the limit $M \to \infty$. Only in this case will stars in a macrocell move differently from one another, allowing us to write schematically

$$\frac{\partial \Psi}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \Psi}{\partial \boldsymbol{x}} - \frac{\partial \Phi}{\partial \boldsymbol{x}} \cdot \frac{\partial \Psi}{\partial \boldsymbol{v}} = \text{differential streaming contributions}\,. \tag{59}$$

The nonvanishing contributions ("diffusion" in both $\boldsymbol{x}$ and $\boldsymbol{v}$) of the right-hand side of equation (59) can then lead to an increase of the entropy associated with Ψ. Clearly, the concept of an irretrievable loss of information which accompanies the passage from a microscopic to a macroscopic level of description becomes a little slippery if the continuum limit is taken *before* making the statistical arguments (cf. Lynden-Bell 1967).

b) *Integrals of Motion*

At this point we should also comment on the issue of an infinity of conserved quantities which Lynden-Bell raises as defect (vi) of his theory (see § 9 of Lynden-Bell 1967). The conserved quantities discussed in Appendix II of Lynden-Bell's paper are implicitly contained in Jeans's theorem which states that the most general solution of equation (57) is given by an arbitrary (single-valued) function of the six independent integrals of the motion (per unit mass) of a star:

$$\psi = \psi(I_1, \ldots, I_6)\,. \tag{60}$$

Jeans's theorem is, of course, the physical basis for our formulation of the microscopic exclusion principle in § III*a*. The notion that the equations of motion of a star do not involve the mass of that star when N is very large has been incorporated by our adoption of the constraints (31) when more than one mass species are present.

Otherwise, a fundamental assumption which underlies our statistical analysis is that *during periods of violent relaxation*, the potential $\phi(\boldsymbol{x}, t)$ allows *none* of the six integrals, $I_1, \ldots, I_6$, to be *isolating* (see, however, Shu 1969 and eq. [62]). (In fact, even when the potential is steady and axisymmetric, usually three of the six integrals can be expressed as phase integrals and are nonisolating.) For statistical purposes, we can ignore the existence of nonisolating integrals of motion. If our neglect of Jeans's theorem were complete, we would be ignoring a conserved function of six independent variables, i.e., a sixfold infinity of conserved quantities (in the continuum limit $N \rightarrow \infty$).

The issue of ignorable integrals of motion is not new to statistical mechanics. Any N-body system, collisionless or not, always has, in principle, $6N$ independently conserved integrals of motion. These $6N$ integrals of motion stand mathematically in the same characteristic relation to Liouville's equation (3) that the six integrals do to Liouville-Boltzmann's equation (46) or (57). (The contraction of $6N$ to 6 is effected by the ansatz [44].) The assumption that none of the six integrals of motion associated with equation (57) is isolating is analogous to the assumption that, except for the energy integral (2), none of the $6N$ integrals of motion associated with equation (3) constrains the long-term motion of a system point in Γ-space. The latter assumption underlies the ergodic hypothesis which equates the time-average behavior of a single system to the ensemble-average behavior of the microcanonical ensemble. The ergodic hypothesis is very plausible, but its rigorous proof is obviously very difficult. (See Uhlenbeck and Ford 1963 for a more detailed discussion.) Let it suffice here to assume that the important conserved quantities can usually be recognized by inspecting the symmetry properties of the governing Hamiltonian.

c) *Incomplete Relaxation*

If violent relaxation could proceed to completion and we were to take the continuum limit, the final distribution would have the form given by equation (37):

$$\Psi = A \exp\left[-c_0^{-2}(\tfrac{1}{2}|\boldsymbol{v}|^2 + \Phi)\right]. \tag{61}$$

When applied to a spherically symmetric stellar system, equation (61) substituted into Poisson's equation leads to the Lane-Emden equation for an isothermal sphere. It is well known that unbounded isothermal spheres of extended spatial structure have infinite masses (see, e.g., Chandrasekhar 1939), whereas isolated isothermal spheres of finite mass have zero spatial structure. Insight into the nature of this difficulty can be gained by examining a sequence of *bounded* isothermal masses of given total energy confined by successively larger reflecting spheres (Antonov 1962; Lynden-Bell and Wood 1968). For small confining volumes and small density contrasts between center and edge, the equilibrium distribution (61) does indeed correspond to a local maximum for the entropy when the total mass and energy are given. But for large confining volumes and large density contrasts, the distribution (61) actually yields a local minimum for the entropy. Such states must therefore be thermodynamically unstable; this fact is a rude reminder that, strictly speaking, our calculations of §§ III and IV yield only *extremum* states for the entropy.

Because of such difficulties, one occasionally reads the opinion that thermodynamics (the continuum limit of statistical mechanics) cannot be applied to self-gravitating systems. However, thermodynamics as an empirical science has survived both the revolutions of quantum mechanics and special relativity, and it is likely that the reconciliation between thermodynamics and gravitation theory will find our concepts about the latter more changed than our concepts about the former. In particular, Lynden-Bell and Wood's (1968) rediscussion of the thermodynamics of stable and unstable bounded isothermal spheres makes a plausible case for the interpretation of the "gravothermal catastrophe" as a phase transition phenomenon. Any further development along these lines (e.g., whether the phase transition is first order or second order; see Callen 1962) would require a more detailed specification of the exact nature of the condensed phase (see, e.g., Lynden-Bell and Lynden-Bell 1977). Massive black holes are currently a popular topic of speculation, and it would be interesting to pursue the relationship of the kinetic theory approaches (e.g., Bahcall and Wolf 1976) to the thermodynamic viewpoint developed by Lynden-Bell and Wood.

In any case, it is clear that the outer parts of observed stellar systems cannot be completely relaxed. Thus, it would be informative to study the functional form of the coarse-grained distribution function which maximizes the entropy (35) when macroscopic constraints additional to equations (36) are applied.

An example of such a study was given by Shu (1969) who maximized the entropy of a flat galaxy under the constraint of the detailed conservation of the angular momentum of each star. The (partially relaxed) equilibrium distribution function in these circumstances is a modified Schwarzschild distribution having many of the properties observed to be present in the distribution of disk stars in the solar neighborhood. Toomre and Mark (1970, private communication) have pointed out that my original distribution function also has a divergent total mass. To repair this deficiency, we can either (1) enclose our system artificially inside a reflecting box, or (2) use the device of "lowering" the modified Schwarzschild distribution in an analogous fashion to King's (1966) use of lowered Maxwellians in his models of spherical stellar systems. To be specific, the lowered modified Schwarzschild distribution is

$$\Psi = P\{\exp[-(\epsilon - \epsilon_c)/c_0^2] - \exp[\epsilon_c/c_0^2]\} \qquad \text{for } \epsilon \leq 0\,, \tag{62a}$$

$$\Psi = 0 \qquad \text{for } \epsilon \geq 0\,, \tag{62b}$$

where ϵ is the energy per unit mass of an actual star, ϵ_c is the energy per unit mass of a hypothetical star in a circular orbit with the same angular momentum per unit mass as the actual star, and P and c_0 are functions of the angular momentum per unit mass.

The distribution (62) can have a finite total mass and yet differ only by an exponentially negligible amount from equation (16) of Shu (1969). (We assume for disk stars that the velocity dispersion c_0 is small in comparison with the circular speed.) The introduction of the cutoff (62), thus, does not modify any of the asymptotic applications that have been used for the modified Schwarzschild distribution (e.g., in density-wave theory). It is vexing but not surprising that such ad hoc procedures have to be applied to the statistical mechanical results to obtain total masses which are not formally divergent. Hopefully, cutoff procedures can be introduced in a more satisfactory manner if one succeeds in formulating a valid kinetic description of the violent relaxation process along the lines discussed in § V*a*.

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REFERENCES

Aarseth, S. J., and Lecar, M. 1975, *Ann. Rev. Astr. Ap.*, **13**, 1.
Antonov, V. A. 1962, *Vestn. Leningrad Gos. Univ.*, **7**, 135.
Bahcall, J. N., and Wolf, R. A. 1976, *Ap. J.*, **209**, 214.
Callen, H. B. 1962, *Thermodynamics* (New York: Wiley).
Chandrasekhar, S. 1939, *Stellar Structure* (Chicago: University of Chicago Press).
Cuperman, S., Harten, A., and Lecar, M. 1971, *Ap. Space Sci.*, **13**, 425.
Gott, J. R. 1977, *Ann. Rev. Astr. Ap.*, **15**, 235.
Hénon, M. 1964, *Ann. d'Ap.*, **27**, 83.
Huang, K. 1963, *Statistical Mechanics* (New York: Wiley).
King, I. R. 1966, *A.J.*, **71**, 64.
Layzer, D. 1967, in *Relativity Theory and Astrophysics*, Vol. **1**, *Relativity and Cosmology*, ed. J. Ehlers (Providence: Am. Math. Soc.), p. 237.
Lynden-Bell, D. 1967, *M.N.R.A.S.*, **136**, 101.
Lynden-Bell, D., and Lynden-Bell, R. M. 1977, *M.N.R.A.S.*, **181**, 405.
Lynden-Bell, D., and Wood, R. 1968, *M.N.R.A.S.*, **138**, 495.
Montgomery, D. C., and Tidman, D. A. 1964, *Plasma Kinetic Theory* (New York: McGraw-Hill).
Shu, F. H. 1969, *Ap. J.*, **158**, 505.
Spitzer, L. 1974, in *IAU Symposium 69*, *Dynamics of Stellar Systems*, ed. A. Hayli (Dordrecht: Reidel), p. 3.
Tolman, R. C. 1938, *The Principles of Statistical Mechanics* (Oxford: Clarendon Press).
Uhlenbeck, G. E., and Ford, G. W. 1963, *Lectures in Statistical Mechanics* (Providence: Amer. Math. Society).
Woltjer, L. 1967, in *Relativity Theory and Astrophysics*, Vol. **2**, *Galactic Structure*, ed. J. Ehlers (Providence: Amer. Math. Society), p. 1.

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