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ON THE STRUCTURE OF CONTACT BINARIES. II. ZERO-AGE MODELS

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ABSTRACT

We construct zero-age models of contact binaries of roughly solar composition with the contact discontinuity hypothesis discussed in an earlier communication. With this formulation, we find it possible to construct systems with common radiative envelopes as well as systems with common convective envelopes. We present explicitly two models with total masses, respectively, of 1.5 $M_\odot$ and 3 $M_\odot$; both binary models have a mass ratio chosen equal to 0.5. The properties of the interior structure of these models are compared with the properties of zero-age single stars which have masses corresponding to the individual components. We also compare the predictions of our theory with the empirical period-color relationship found by Eggen for W Ursae Majoris stars. The agreement with observations is satisfactory.

Subject headings: stars: binaries — stars: W Ursae Majoris

I. INTRODUCTION

W Ursae Majoris stars pose intriguing problems for the theory of stellar interiors. Kraft (1967) pointed out that, among all eclipsing binaries, W Ursae Majoris stars have a virtual monopoly on the short periods, and he gave a cogent presentation of the statistical arguments which favor a main-sequence interpretation for these contact binaries. Following a similar line of reasoning, Lucy (1968) concluded that W Ursae Majoris stars "must themselves exist at age zero and must be presently evolving on a nuclear time scale." Lucy went on to construct zero-age contact binaries based on his elegant idea that the common envelope must be convective with a single value for the specific entropy. Unfortunately, despite the stimulation of much related work (Moss and Whelan 1970; Hazlehurst 1970; Biermann and Thomas 1972; Moss and Whelan 1973; Ruciński 1974; and references therein) and despite some promising beginnings—notably with respect to the synthesis of realistic light curves—this line of research has failed to produce stellar models which correspond to observations (Lucy 1976).

Lucy himself has now abandoned his original position on the stability of zero-age contact binaries and has adopted, instead, a complicated scenario involving, for W-type systems, thermal relaxation oscillations between a semidetached state and a state of marginal contact, and, for A-type systems, some core evolution (see also Flannery 1976).

We find this proposal to be implausible on three counts. First, we find it difficult to conceive of *oscillations* about a *nonexistent* equilibrium state. For oscillations to exist in a closed mechanical system, an equilibrium state must almost certainly exist in the middle, although it may turn out that such an equilibrium is indeed overstable. Thus we regard it as more profitable to look first for possible states of equilibria and to study later the question of their stability. Second, we share Lucy's worry that an oscillatory model will produce non–EW light curves about half of the time. This tendency would conflict with the observed statistics of the light curves of eclipsing binaries which have periods shorter than about half a day (Kraft 1967; Lucy 1976). Third, we disagree with the *a priori* assumption which is the prevailing opinion in this field, that there is a fundamental distinction between common envelopes which are convective and common envelopes which are radiative.

Let us elaborate on the last point because it is related to the fatal flaw of Lucy's original models. If we assume that a contact binary is in mechanical equilibrium and thermal equilibrium, the barotropic condition and standard mixing-length theory applied to convection zones show that the convective flux is uniform on equipotential surfaces (see eq. [5] of Anderson and Shu 1977). If we further assume that the contact binary has a common convective envelope whose properties are continuous across the inner critical surface, we can prove, as Lucy (1968) did to derive equation (13) of his paper, that

$$L_{\rm II}/L_{\rm I} = A_{\rm II}/A_{\rm I}\,, \tag{1}$$

where $L_{\rm I}$ and $L_{\rm II}$ are the rates of energy generation in the two interiors and $A_{\rm I}$ and $A_{\rm II}$ are the areas of the two Roche lobes. Because of mass conservation, meridional circulation of the standard type does not carry any net energy across a closed equipotential surface (see, e.g., Strittmatter 1969); therefore such circulation does not affect the equality expressed above.[1] Difficulty arises because equation (1) cannot

[1] This statement ignores quadratic effects, where by zero-order accuracy we mean the barotropic state: $P = P(\Phi)$, $\rho = \rho(\Phi)$, and $T = T(\Phi)$. Correlations do exist between the velocity field and state-variable fluctuations produced by meridional circulation, and these correlations will lead to a net transport of heat across a closed equipotential surface.

be satisfied in general unless the two component stars are identical. (Indeed, suppose physical properties to be continuous across the Roche lobes. Then it has been explicitly demonstrated that, on a thermal time scale, a contact binary with a common convective envelope either will evolve to unit mass ratio [Hazlehurst and Meyer-Hofmeister 1973], or will try to break contact [Lucy 1976; Flannery 1976].) This amplification of Kuiper's (1941) paradox for systems with common convective envelopes stands on an equal footing with Lucy's amplification for systems with common radiative envelopes. It does not help the *thermal transport* considerations to say, as Lucy did, that *adiabaticity* is the usual zeroth order approximation for stellar convection zones any more than it would to say that isotropy and *isothermality* are the usual zeroth order approximation for the radiation field in stellar interiors (see Schwarzschild 1958, pp. 39–42).

Notice that it also does not help to assume the systems to be evolved, because the proofs do not depend on the chemical composition being homogeneous, only on its being uniform on equipotential surfaces in the common envelope. To escape the imposition of an extra constraint of the type given by equation (1), either we must give up thermal equilibrium, or we must give up continuity of physical properties across the inner critical surface. Lucy (1976) and Flannery (1976) choose the first alternative; we choose the second (Shu, Lubow, and Anderson 1976, hereafter Paper I).

In this paper, we shall use the formalism developed in Paper I to construct explicitly two equilibrium models of zero-age contact binaries: one has a common convective envelope; the other has a common radiative envelope. The mass ratio (0.5), the total mass, and the separation of the individual components which are chosen have the result that the first model has a period which is comparable to the shortest observed, and the second model has a spectral type which is comparable to the earliest observed for W Ursae Majoris stars. To be able to construct these equilibrium models, we require only the degree of freedom allowed by the existence of a contact discontinuity between the base of the common envelope and the Roche lobe of the cooler star. Paper I argued that such a discontinuity could be maintained by fluid flow against thermal diffusion if the dynamical time scale is locally a small fraction δ of the thermal time scale at the base of the common envelope. In this paper we verify the smallness of δ for our models *a posteriori*. In a parallel paper (Anderson and Shu 1977), we develop the implications of a very small value of δ for the light curves of W Ursae Majoris stars.

Before we proceed, however, to construct our interior models, we comment on the claim that W-type systems are only marginally in contact (Lucy 1973; Ruciński 1973). The work of Lucy (1976) and Flannery (1976) is motivated at least in part by this claim. The tendency toward marginal contact, if present, receives no immediate explanation within the context of our theory; we can accommodate small filling factors, f, but nothing in the present development of our theory suggests that the distribution of possible f-values should be especially skewed in the interval (0, 1). We suggest a reanalysis of the light curves that uses the theory of Anderson and Shu (1977).

II. INPUT PHYSICS

The governing equations, boundary conditions, and jump conditions for our problem have been recorded in Paper I. Henceforth, we shall refer to the equations of Paper I with the prefix I followed by a decimal point and the appropriate equation number. For the present, we merely remark that once the chemical composition and constitutive relations ("input physics") have been specified, our formulation yields a unique prescription for the construction of the model system given the masses, $M_{\rm I}$ and $M_{\rm II}$, of the two individual components and the separation D of their centers. We specify below our choices for the input physics.

a) Chemical Composition and Equation of State

We are interested in homogeneous stars of disk population abundances (see Kraft 1965 for a discussion of the spatial distribution of W Ursae Majoris stars in the Galaxy). For sake of definiteness, we choose $X = 0.70$, $Y = 0.28$, and $Z = 0.02$, corresponding to a mean molecular weight for the interior of $\mu = 0.6173$. Since we deal primarily with systems of moderate masses, we ignore the role of electron degeneracy. Thus we may write the equation of state as

$$P = \frac{\rho k T}{\mu m_{\rm H}} + \frac{1}{3} a T^4 . \tag{2}$$

b) Opacity and Energy-Generation Rate

We follow Iben and Ehrman (1962) and approximate the opacity and energy-generation rate by the formulae

$$\kappa = 0.19(1 + X) + \kappa' ,$$

$$\log_{10} \kappa' = G + \alpha' \log_{10}\left[\frac{\rho}{2}(1 + x)\right] - F \log_{10} T_6 , \tag{3a}$$

$$\epsilon = \rho[\epsilon_1 \phi(X, Z, T) X^2 T_6{}^4 + \epsilon_2 X Z T_6{}^{18} \times \exp(1.81 \rho^{1/2}/T_6{}^{3/2})] , \tag{3b}$$

where the functions G, α', F, ϕ and the constants ϵ_1, ϵ_2 are defined on pp. 771–773 of their article. The first term in equation (3a) represents the effect of electron scattering; the second term is the analytic fit of Iben and Ehrman to the opacity tables of Keller and Meyerott (1955). The first term in equation (3b) gives the energy-generation rate by the p-p chain; the second, by the CN cycle.

c) *Adiabatic Constant for Convective Envelopes*

Some of our models will have common convective envelopes, and the interior structure will be influenced by the value of the specific entropy reached in the subsurface layers. Below these layers, the pressure varies with temperature in accordance with the adiabatic relation for a perfect gas:

$$P = KT^{5/2}\,, \tag{4}$$

where K is the adiabatic constant. It is conventional to regard the determination of the value of K as an outer boundary condition for the interior integration.

In accordance with Lucy (1968), we adopt the following procedure to determine K. We replace the very thin superadiabatic region, which contains the photosphere at the top, with an equivalent plane-parallel slab. The gravity of this plane-parallel region is taken to be equal to the average surface gravity, $\bar{g}$, of the modified Roche lobe model (see eqs. [I.11] and [I.16]), and the effective temperature, equal to the average effective temperature, $\bar{T}_e$, of the system. The latter is defined via

$$\sigma\bar{T}_e^{\,4} = (L_{\rm I} + L_{\rm II})/A\,, \tag{5}$$

where $L_{\rm I} + L_{\rm II}$ is the sum of the luminosities generated in the two stellar cores and A is the total surface area of the contact binary. Under circumstances which are characteristic of the outer convection zones of dwarf stars, it is well known that, as we move from the photospheric layers toward the deeper layers, the value of $PT^{-5/2}$ asymptotically reaches a constant value K which depends only on $\bar{g}$ and $\bar{T}_e$ (once the chemical composition is specified). In fact, Lucy (1967) has noted that K can be approximated as a function only of the combination of variables[2]

$$\xi \equiv \bar{T}_e\bar{g}^{-0.08}\,. \tag{6}$$

Figure 1 shows our adopted relationship for $K = K(\xi)$. The form of this curve follows from the calculations of Baker and Temesváry (1966) for a ratio of mixing length to pressure scale height equal to unity.

We remark that the exact dependences of K on $\bar{g}$ and $\bar{T}_e$ are not essential to the fundamental development of our theory. For the problem of stellar interiors, we can regard Figure 1 merely as a convenient replacement for an accurate calculation of convective transport in the outer envelope. The possible future emergence of such an accurate calculation will improve our quantitative results, but it should not modify any of our qualitative conclusions.

III. RESULTS

a) *Accuracy of Calculational Technique*

The main computational techniques in our formulation of the problem (Paper I) are (*a*) to use the effective potential Φ as the relevant "coordinate" variable to integrate the structure equations, and (*b*) to approximate Φ as a function of spatial location by the effective potential associated with two *spherical* mass distributions which revolve in a circular orbit about one another at a distance D apart ("modified Roche potential"). To obtain the optimum "modified Roche potential," we have adjusted each spherical mass distribution to have ultimately the same mass within a given volume interior to a sphere as we obtained from the structure calculation for the mass within a corresponding volume interior to a nonspherical equipotential surface. Clearly, this "best spherical approximation" for the potential calculation can be obtained only in an iterative sense. Convergent schemes for the entire procedure can be invented with some trial and error (Lubow 1977), but details will not be given here. The similarity of our approach to the central field approximation used in quantum mechanics to calculate the structure of multielectron atoms (Schiff 1955) leads us to call our formulation the bicentral field (BCF) approximation.

As a check on the accuracy of the BCF procedure, which alternates between a potential calculation and a structure calculation, we used the same iterative method to compute the properties of a 1 $M_\odot$ star when it is accompanied by a 0.5 $M_\odot$ star orbiting at a distance equal to 10^5 unperturbed radii of the 1 $M_\odot$ star. We then compared the "converged" solution with the result of a conventional single-star calculation which used the same input physics (§ II). Since the influence of the companion star is entirely negligible for such a wide binary, the differences between the results of the conventional calculation and the BCF calculation reflect solely on the accuracy of the numerical procedure. We found the numerical errors in our BCF calculation not to exceed ~0.3%. The quoted results which follow for contact binaries are internally accurate to about this level of precision. It is more difficult to estimate the intrinsic error associated with the BCF approximation, but judging from Mark's (1968) analogous experience with uniformly (and nonuniformly) rotating stars, we guess that the error is likely to be quite small.

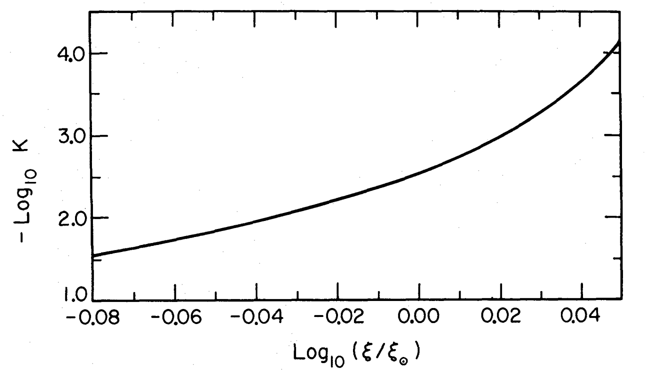

FIG. 1.—The adopted relationship giving the adiabatic constant K for surface convection zones as a function of the parameter $\xi = \bar{T}_e\bar{g}^{-0.08}$.

[2] Notice that g and T_e enter in eq. (6) only in terms of their average values. We do not use Fig. 1 in the differential sense that Lucy does to obtain his law of "convective gravity brightening" (see also Anderson and Shu 1977).

TABLE 1

ZAMS STARS WITH $X = 0.70$, $Y = 0.28$, $Z = 0.02$

PARAMETERS	SINGLE STARS			CONTACT BINARIES	
	$2\,M_\odot$	$1\,M_\odot$	$0.5\,M_\odot$	$2\,M_\odot + 1\,M_\odot$	$1\,M_\odot + 0.5\,M_\odot$
Luminosity	7.70×10^{34}	3.34×10^{33}	1.83×10^{32}	$7.51 \times 10^{34} + 3.21 \times 10^{33}$	$3.13 \times 10^{33} + 0.817 \times 10^{32}$
Radius	9.33×10^{10}	6.35×10^{10}	2.99×10^{10}	$D = 19.5 \times 10^{10}$	$D = 12.5 \times 10^{10}$
ρ_c	69.5	95.4	71.7	69.9, 94.9	94.7, 63.8
T_c	2.19×10^7	1.44×10^7	8.84×10^6	2.18×10^7, 1.43×10^7	1.42×10^7, 7.65×10^6
P_c	2.04×10^{17}	1.83×10^{17}	8.47×10^{16}	2.05×10^{17}, 1.81×10^{17}	1.80×10^{17}, 6.53×10^{16}
Envelope K	Radiative envelope	2.96×10^{-3}	0.240*	Common radiative envelope	1.09×10^{-2} (Common convective envelope)
T above Roche lobe	...	...	...	5.70×10^5	3.90×10^5
T below Roche lobe	...	...	...	5.70×10^5, 1.82×10^5	3.90×10^5, 1.74×10^5
P at Roche lobes	...	...	...	5.55×10^{10}	1.05×10^{12}
δ at base of common envelope	...	...	...	3×10^{-7}	2×10^{-9}
f and period	...	...	...	$f = 0.84$, $P_d = 0.314$ d	$f = 0.41$, $P_d = 0.228$ d
$\log_{10} \bar{g}$ at surface	4.484	4.517	4.871	4.364	4.526
$\overline{T}_e$	10600	5840	4120	9240	5380

* Value not obtained from Fig. 1 but chosen empirically.

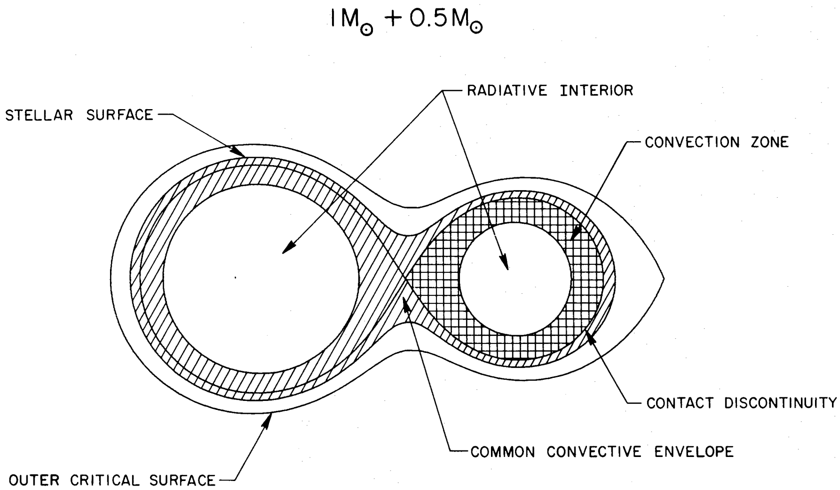

FIG. 2.—Equatorial section of the interior structure of a 1 $M_\odot$ + 0.5 $M_\odot$ zero-age contact binary of solar composition. The filling factor of this model is $f = 0.41$, and the binary period is $P_a = 0.228$ d.

b) Two Models of Zero-Age Contact Binaries

Table 1 gives a comparison of the properties of three zero-age main-sequence (ZAMS) single stars of masses 2 $M_\odot$, 1 $M_\odot$, and 0.5 $M_\odot$, with the properties of two *converged* BCF models of contact binaries with masses 2 $M_\odot$ + 1 $M_\odot$ and 1 $M_\odot$ + 0.5 $M_\odot$. The stars in the latter two models are placed at distances D apart, so that the filling factors are, respectively, 0.84 and 0.41, and the periods are 0.314 d and 0.228 d. The filling factor f is defined by us to be

$$f = (\Phi_s - \Phi_1)/(\Phi_2 - \Phi_1)\,, \tag{7}$$

with $f = 0$ for marginal contact, while $f = 1$ for full contact. Figures 2 and 3 give scaled representations of the same contact binary models.

The crucial feature of both models is the contact discontinuity at the Roche lobe of the cooler star. This contact discontinuity is envisioned as being maintained against thermal diffusion by the sequence of events described in § III of Paper I. In particular, to relieve itself of the energy generated in its core, the smaller star transfers heat to its larger companion in the form of a very slow mass flow through a small neighborhood of the inner Lagrangian point, L_1. In a steady state this mass loss is recovered by a net diffusive conversion of hot gas characteristic of the material above the Roche lobe of the smaller star into cool gas characteristic of the material beneath it. This diffusive transformation of hot gas into cool gas occurs because the radiative plus convective heat flux entering the transition layer from below (typically, $L_{\rm II}/A_{\rm II}$) is less than the heat flux leaving from above (typically,

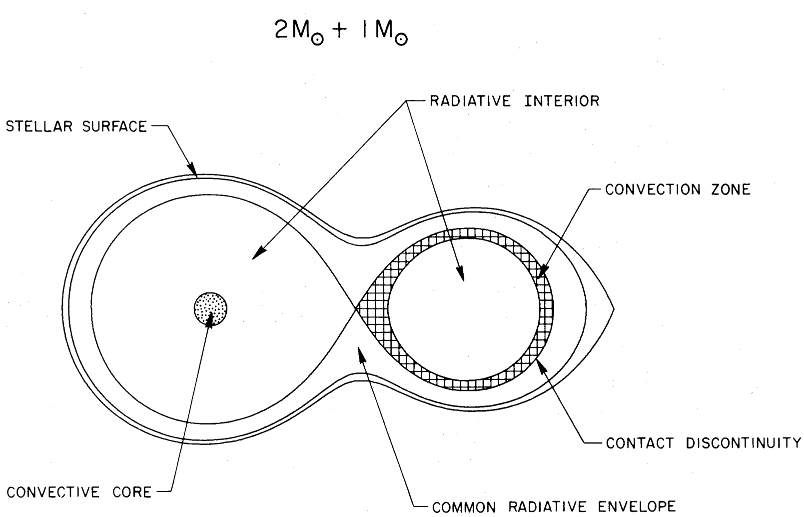

FIG. 3.—Equatorial section of the interior structure of a 2 $M_\odot$ + 1 $M_\odot$ zero-age contact binary of solar composition. The filling factor of this model is $f = 0.84$, and the binary period is $P_a = 0.314$ d.

$[L_{\rm I} + L_{\rm II}]/[A_{\rm I} + A_{\rm II}]$). Presumably, the condition that the diffusive conversion exactly balance the hydrodynamic mass loss through the L_1 region in a steady state is met by the adjustment of the thickness and the structure of the actual transition layer. We expect the transition layer to be very thin if the ratio of the actual heat flux to the hypothetical heat flux which could be carried if the matter were to move at the speed of sound, a_s, is a very small number at the base of the common envelope:

$$\delta \equiv F/\rho h a_s \ll 1\,. \qquad (8)$$

Since $\delta = 2 \times 10^{-9}$ in our $1\ M_\odot + 0.5\ M_\odot$ model and $\delta = 3 \times 10^{-7}$ in our $2\ M_\odot + 1\ M_\odot$ model, we can justify our treatment of the transition layers as contact discontinuities *a posteriori*.

As explained in Paper I, enormous simplicity is gained by the approximation of the actual transition layer as a contact discontinuity: we do not then need to calculate the complex fluid flow described above to be able to calculate accurately the structure of the system.[3] We give below a detailed description of the properties of the two models calculated by using this procedure.

i) *The 1 $M_\odot$ + 0.5 $M_\odot$ Model*

This model is close to one extreme of W Ursae Majoris stars in terms of the shortness of its period. Our model has a common convective envelope with a K-value which is between the K-values of the unperturbed $1\ M_\odot$ and $0.5\ M_\odot$ stars. Figure 2 shows the convective regions beneath the Roche lobes of both components to be fairly deep.

From Table 1 we see that the structure of the $1\ M_\odot$ component is relatively unaffected by the presence of its $0.5\ M_\odot$ companion. Some distortion of the superficial layers occurs, but the major effect on the interior is the slight decrease of the luminosity generated in its core: from 3.34×10^{33} ergs s^{-1} to 3.13×10^{33} ergs s^{-1}. This 6% decrease can almost be wholly associated with the rotational lowering of the effective gravity in the central regions because the two components are assumed to spin synchronously with the orbital motion (cf. Kippenhahn and Thomas 1970; Faulkner, Roxburgh, and Strittmatter 1968). The changes of other physical qualities, such as the central temperature, also coincide with Mark's analysis of the role of uniform rotation.

The roughly twofold decrease in the energy-generation rate of the $0.5\ M_\odot$ component cannot be explained in the same manner; we estimate that rotation accounts for only 9% of the decrease. The primary contribution to changing the interior structure of the $0.5\ M_\odot$ component is the "bottling up" of its convective energy-transfer mechanism at the Roche surface. The $0.5\ M_\odot$ star swells up from its unperturbed size to fill its Roche lobe (the amount of mass in the common envelope is very small and is divided in rough proportion to the unperturbed masses). It accomplishes this by making its outer convection zone (beneath the Roche surface) less extensive (involving a reduction of the K-value or an increase of the specific entropy). This process is accompanied by a substantial reduction of the central temperature, which leads to a much reduced energy-generation rate.[4]

To summarize, being in contact with a hotter star raises the temperature of the outer regions of the cooler star. This process expands the latter, making the central temperature of the cool star even cooler!

ii) *The 2 $M_\odot$ + 1 $M_\odot$ Model*

Our ZAMS model for a $2\ M_\odot + 1\ M_\odot$ system shows far fewer dramatic changes of the interior structure from the corresponding single stars than the previous example. This state of affairs depends on the unperturbed mass-volume relation for the mass range which borders the crossover from dominant energy generation by the *p-p* chain to dominant energy generation by the CN cycle. The ratio of the unperturbed volumes of two ZAMS single stars on either side of the crossover mass (roughly $1.9\ M_\odot$) turns out coincidentally to be nearly the same as the ratio of the volumes of the Roche lobes corresponding to these two masses.

The above coincidence accounts in part for Lucy's (1968) ability to construct zero-age contact binary models with *shallow* common envelopes, without having to introduce a contact discontinuity. Of course, the departure from an *exact* coincidence limits Lucy to an eigenvalue solution which can be satisfied only by adopting extreme Population I abundances (see also Moss and Whelan 1970). We are able to adopt more conventional abundances and can construct models over a much larger range of mass ratios and total masses because (*a*) we have the degree of freedom associated with the contact discontinuity which offsets the overspecification of the boundary conditions, and (*b*) we have the finite margin of a variable filling factor. In any case, different modes of energy generation in the two cores do *not* constitute a fundamental aspect of our models; we can accommodate them, but we do not require them.

In the specific $2\ M_\odot + 1\ M_\odot$ model which is presented, the major changes from the structure of unperturbed single stars occur in the outer layers. In particular, the system radiates the combined luminosity, $L_{\rm I} + L_{\rm II} = 7.83 \times 10^{34}$ ergs s^{-1}, through a

[3] The situation is comparable to the role played by convection in the theory of stellar interiors. When convection is present, we do not need to know the precise details of the convective flow to arrive at the conclusion that the temperature gradient must be nearly the adiabatic gradient. Indeed, mixing-length theory states that the superadiabatic temperature gradient is very small—of order $\delta^{2/3}$; thus, the *structure* (but not the convective flux) of convection zones in the *interiors* (but not the outermost layers) can be calculated accurately by simply approximating the actual gradient by the adiabatic gradient. Again, a complex flow calculation can be bypassed because $\delta \ll 1$.

[4] At the decreased central temperature, the central pressure of the $0.5\ M_\odot$ component is missing about a 3% contribution from electron degeneracy.

common surface area, $A = 1.89 \times 10^{23}$ cm^2, instead of the individual unperturbed luminosities, 7.70×10^{34} ergs s^{-1} and 3.34×10^{33} ergs s^{-1}, through the individual unperturbed areas, 1.09×10^{23} cm^2 and 5.07×10^{22} cm^2. The resulting average effective temperature for the contact binary is $\bar{T}_e = 9240$ K, and the system has a common *radiative* envelope. (The region just below the Roche surface of the 1 $M_\odot$ star is convective; see Fig. 3.) For the dwarflike average surface gravity of our contact binary, the spectral type is roughly A2. Such a spectral type is close to the earliest yet observed for stars of the W Ursae Majoris type.

IV. COMPARISONS WITH OBSERVATIONS

It has become standard practice since Lucy's (1968) pioneering work to compare the predictions of models of the interior structure of contact binaries with the empirical period-color relation found by Eggen (1961, 1967) for W Ursae Majoris stars. Our theoretical models of contact binaries, even at zero age and given chemical composition, are characterized not by a single free parameter but by three: the masses of the two stars and the separation distance D (or filling factor f). Thus we do not expect, theoretically, a *unique* period-color relation, only a period-color correlation (in the obvious sense that shorter period systems *tend* to be redder). Of course, if we fix two quantities—say the mass ratio $q = M_{II}/M_I$ and the filling factor f—we will obtain a single theoretical curve in the plane, log $\bar{T}_e$ versus log P_d. Different points on the curve correspond to different total masses. If we now allow the filling factor f to vary arbitrarily, this single curve broadens into a narrow band. The shaded band in Figure 4 shows our estimate of the ZAMS band for the mass ratio $q = 0.5$ and the chemical composition $X = 0.70$, $Y = 0.28$, and $Z = 0.02$. For reference we have plotted the two specific models discussed in § III as X's labeled 1 + 0.5 and 2 + 1.

Also plotted in Figure 4 as a dotted line labeled TAMS is our tentative estimate for the period-effective temperature relation of (half-filled) contact binaries where the primary component has terminated its main-sequence phase of stellar evolution in the sense defined by Iben (1967). The question marks at the ends of this dashed line refer to the very crude nature of this estimate, which serves here only as an intuitive replacement for future detailed calculations. The estimate is made on the basis of the following line of reasoning.

To end with as wide a separation as implied by the position of the TAMS line, the two component stars must almost certainly begin their ZAMS phase as detached binaries. When the larger star reaches the end of its main-sequence phase, it swells up to fill its Roche lobe and begins to transfer mass to its essentially unevolved companion in the case A of mass transfer (Plavec 1968: Paczyński 1971). According to the calculations of Benson (1970), this semidetached state is very short-lived and quickly results in the two stars coming into contact after only a small amount of mass transfer. Although the fate of the system after first contact has been made is uncertain (see, e.g., Flannery and Ulrich 1977), we believe that a common envelope develops which eventually becomes dense enough to provide the necessary back-pressure to stop the mass transfer on a rapid time scale. Thus, after a period of readjustment, the system will settle down to an *equilibrium* contact binary which is the evolved counterpart of the systems computed in the previous

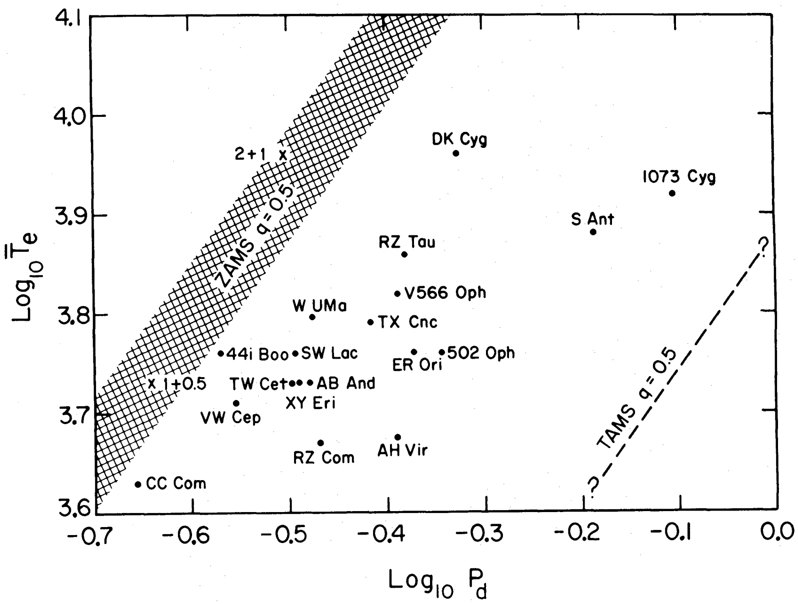

FIG. 4.—The period-effective temperature relationship for main-sequence contact binaries. Periods are measured in days and temperature in kelvins.

section. These evolved systems must also contain a contact discontinuity, and their subsequent evolution would presumably take place on a nuclear time scale (until other catastrophes strike!). In our calculations of ZAMS contact binaries, we have found that the total volumes and total luminosities of the contact system are not very different from the sum of the volumes and luminosities of the individual unperturbed stars. If a similar result is boldly hypothesized for TAMS contact binaries, we can guess the periods and effective temperatures by adding the volumes and luminosities of Iben's (1967) calculations for single stars—say, a TAMS 1 $M_\odot$ star and a ZAMS 0.5 $M_\odot$ star—and require that this volume be occupied and luminosity be radiated in a (half-filled) Roche geometry. The dashed line labeled TAMS is obtained by drawing a straight line through such a computed point parallel to our ZAMS band. Clearly, the TAMS calculation is very rough, and we do not expect it to give more than an approximate indication of the true state of affairs.

Finally, we plot as individual points in Figure 4 all the observed W Ursae Majoris stars which have spectroscopic mass ratios between 0.33 and 0.67. (Our list comes from Binnendijk 1970 and Ruciński 1974.) We have also included CC Com, the spectroscopic mass ratio of which is not known at the time of the writing of this paper, but whose mass ratio, effective temperature, and period have been determined from an extensive analysis of the photometric light curve (Ruciński 1976). The effective temperatures of the other systems are obtained by converting the spectral type given by Binnendijk (1970) with the tables in Allen (1955) for dwarf stars. The use, instead, of Eggen's (1961, 1967) $B - V$ colors would result in some differences, but they are inconsequential for our purposes.

It can be seen that all of the observed systems fall between the ZAMS band and the TAMS line. Most of the systems lie closer to the ZAMS band than to the TAMS line. In general, Figure 4 supports Kraft's (1967) opinion that most of the short-period W Ursae Majoris stars were born in a contact configuration or close to it, while some of the longer-period systems may have evolved to this state. It is somewhat disturbing that no observed system actually falls within our computed ZAMS band, although CC Com and 44i Boo—two lightweight systems—come very close. On the other hand, it is comforting to find that no observed system falls in the forbidden region above and to the left of our ZAMS band. We also point out that our diagram contains not only information concerning periods and effective temperatures but also information concerning mass ratios and total masses. In particular, the spectroscopically and photometrically determined total masses of the observed systems are in approximate accord with their positions relative to our computed points labeled 1 + 0.5 and 2 + 1.

The system TX Cnc is generally regarded as an unevolved W Ursae Majoris system because it is located in the cluster Praesepe and its position in the H-R diagram of that cluster is well below the turnoff point (Kraft 1967). More specifically, the age of the Praesepe cluster is estimated to be 4×10^8 yr, which is only $\frac{1}{4}$ of the main-sequence lifetime of the primary component of TX Cnc if it has a mass of about 1.6 $M_\odot$ and if it undergoes normal stellar evolution. One's impression of the location of TX Cnc in Figure 3 suggests that it should be more than a quarter of the way through its main-sequence evolution, and this discrepancy may indicate some potential difficulties for our theory. However, the discrepancy is not larger than the various theoretical and observational uncertainties; moreover, it would be difficult in anybody's theory to explain the position of TX Cnc *with respect to the other observed systems* if it is truly a totally unevolved contact binary. As usual, more observations and more theoretical developments are needed to clarify the issues.

V. DISCUSSION

In this paper we have explicitly computed zero-age models for contact binaries along the lines proposed in Paper I. In particular, we have demonstrated that the introduction of a contact discontinuity at the Roche lobe of the smaller star is crucial to the construction of equilibrium systems. The ZAMS models so obtained predict period-effective temperature correlations which are in rough accord with observed W Ursae Majoris stars (Fig. 4). The main discrepancy is that the observed systems tend to have longer periods at a given effective temperature than our zero-age models. At the upper total-mass end, some core evolution of the observed systems within the main sequence can remove the discrepancy. At the lower total-mass end, we may have to invoke the interaction between convection and faster than normal stellar rotation to shift the theoretical ZAMS band toward longer periods (see also Anderson and Shu 1977).

Although the precise nature of the interaction between convection and rotation under astrophysical conditions is not well understood (Durney 1971; Spiegel 1972; Gilman 1974), all workers are agreed that rotation tends to suppress the *efficiency* of convection. Within the context of mixing-length theory, we may attempt to take account of this effect by reducing the ratio of the mixing length to the pressure scale height. This procedure would have the net effect of making the stellar models more radiative and therefore of increasing their volumes (see Schwarzschild 1958, chap. 4). For given total mass, mass ratio, and filling factor, our models of zero-age contact binaries with common convective envelopes would then have longer periods than those used to construct the lower portion of the theoretical ZAMS band of Figure 3.

In any case, the apparent discrepancy between theory and observations evident in Figure 3 is not large, and we are optimistic that slight refinements of the input physics or a slight relaxation of the interpretation that W Ursae Majoris stars are exactly zero age would lead to virtual accord between our theory and the observations. Given this optimistic view of the contact discontinuity model, we should now consider the future avenues of research opened up by our work.

First, a more detailed examination should probably be made of the nature and effect of the flow which provides the redistribution of heat in the "mixing region" (see § II of Paper I). The combination of fluid flow and thermal diffusion that establishes the structure of the contact discontinuity could also be profitably studied.

Second, an investigation should now be undertaken of the point of view expressed in § IV of Paper I, that a contact binary should be considered as a single body which happens to have two centers of concentration of mass, rather than as two separate stars which happen to touch. This study should, in particular, be directed to the question of the stability of our equilibrium models to changes of the mass ratio. We believe that considerable insight may be gained concerning the problem of *secular* instability by examining a sequence of ZAMS models which have the same total mass and the same total angular momentum but different mass ratios. Clearly, the model which has least total energy (or equivalently, as the virial theorem shows, least potential energy) in such a sequence must be secularly stable. Simple arguments suggest that minimizing the interaction energy of the two mass concentrations (while conserving total angular momentum and mass) favors a unit mass ratio, whereas minimizing the sum of the self-energies favors a zero mass ratio. The compromise between these two tendencies may explain why W Ursae Majoris stars have the mass ratios that they do.

Finally, there is the important problem of the evolution of W Ursae Majoris stars (Kraft 1965). Schwarzschild (1958) has already pointed out that meridional circulation may play a very important role in mixing W Ursae Majoris stars (see also Flannery 1976). A second process may also complicate evolutionary calculations, namely, the possibility of mass and angular momentum loss from the system when the common surface expands beyond the L_2 point (Kuiper 1941). The dynamic problem associated with this latter process will be the subject of Paper III of this series (Lubow and Shu 1977).

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