THE ASTROPHYSICAL JOURNAL, 214:798–808, 1977 June 15

ON THE LIGHT CURVES OF W URSAE MAJORIS STARS

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Received 1976 September 16; revised 1976 November 29

ABSTRACT

We develop a physical theory for the light curves of contact binaries based on the assumption that the dynamical time scale is very short in comparison with the thermal time scale at the base of the common envelope. In contrast with the case for common radiative envelopes, the flux distribution in common convective envelopes does not exhibit any effect of gravity brightening. Combined with a unified treatment of reflection and limb darkening, this result produces W-type light curves for W UMa stars of spectral type later than F5 if the orbit is inclined by less than approximately 70°–90°. The sign of the effect is in rough accord with the observations, but some discrepancy remains concerning the magnitude of the effect. We speculate that the interaction between rapid rotation and convection may contribute to the remaining discrepancy; it may also produce the asymmetry and time variability which are observed in some light curves.

Subject headings: stars: binaries — stars: W Ursae Majoris

I. INTRODUCTION

It is now generally accepted that the main features of the light curves of W UMa stars are explained by their being contact binaries with nearly equal effective temperatures for both stellar components (see Osaki 1965; Lucy 1968*b*; Mochnacki and Doughty 1972*a*, *b*; Wilson and Devinney 1973; Ruciński 1974). Less well recognized is that these features are *virtually independent* of any particular theory for the interior structure, but depend only on the base of the common envelope being deep enough so that the regions above it are barotropic (equal pressures, densities, and temperatures on equipotentials) and not baroclinic (surfaces of constant pressure inclined to surfaces of constant density). For distorted envelopes it is well known (von Zeipel 1924; Eddington 1926) that the barotropic tendency is incompatible with the tendency for detailed thermal balance, $\nabla \cdot \boldsymbol{F} = 0$, where $\boldsymbol{F}$ is the total diffusive heat flux (radiative plus convective). Which tendency wins out in the lowest order of approximation depends on the ratio of the dynamical time scale, $t_{\rm dyn}$, to the thermal time scale, $t_{\rm th}$.

To make the pressure uniform on an equipotential surface of horizontal scale length D (equal, say, to the binary orbit separation) generally requires a time $t_{\rm dyn} \sim D/a_s$, where a_s is the speed of sound. To erase a nonzero divergence of the heat flux by diffusive processes at a depth H below the photosphere generally requires a time $t_{\rm th} \sim \rho h H/F$, where ρh is the enthalpy per unit volume. The ratio $t_{\rm dyn}/t_{\rm th}$ can thus be estimated as

$$t_{\rm dyn}/t_{\rm th} \sim \delta D/H\,, \tag{1}$$

where

$$\delta \equiv F/\rho h a_s \tag{2}$$

is the local ratio of the actual heat flux, F, to the hypothetical heat flux, $\rho h a_s$, which could be carried if material were to move at the speed of sound. Notice that δ measures $t_{\rm dyn}/t_{\rm th}$ if the horizontal and vertical length scales, D and H, are comparable.

To fix ideas, we have plotted in Figure 1 the quantity $\log_{10}(\delta D/H)$ against $\log_{10}(H/D)$ for the model of Baker and Temesváry (1966) of the solar convection zone when we take $D = 2\,R_\odot$. The photospheric layers (characterized by $\log_{10}\tau \sim 0$ in the upper horizontal axis) and the superadiabatic part of the convection zone (characterized by where $\log_{10}[(\nabla - \nabla_a)/\nabla_a] \sim 0$) correspond to layers for which $\delta D/H \gg 1$. Thus, if these outermost layers were placed in a contact binary configuration, they would maintain detailed thermal balance—diffusively transporting to above whatever heat they receive from below. Any resulting mechanical disequilibrium in the horizontal directions arising from a baroclinic condition in these layers is unimportant because the ensuing flow leads to a negligible horizontal redistribution of heat *even if the flow were to proceed at the speed of sound.*[1] Clearly, then, the

[1] As a by-product of this argument, we can rule out on *theoretical* grounds models of contact binary envelopes which rely on significant energy transfer between the two stellar components in the superadiabatic part of the common envelope (Whelan 1972). The crucial point is that superadiabatic temperature gradients occur where $\delta \sim 1$ (see eq. [5]); hence, the parameter $\delta D/H$ is large to the extent that H is a small fraction of D; i.e., at these depths, fluid flow finds it difficult to transport heat vertically, much less horizontally.

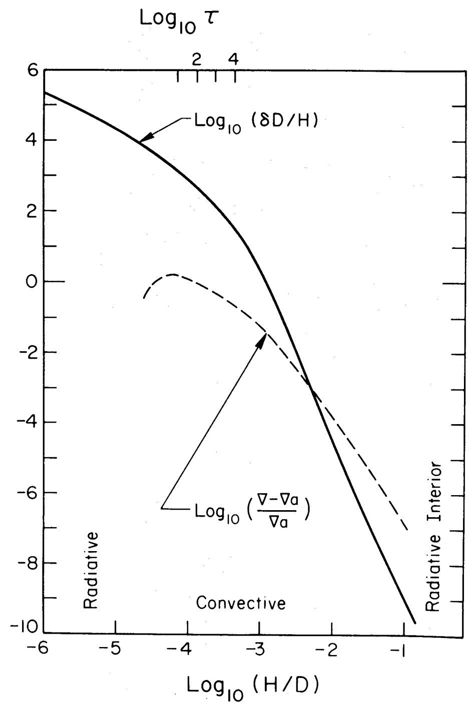

FIG. 1.—The ratio of the dynamical time scale to the thermal time scale, $\delta D/H$, and the fractional excess of the superadiabatic temperature gradient, $(\nabla - \nabla_a)/\nabla_a$, versus the fractional depth, H/D, below the photosphere in the solar convection zone as computed by Baker and Temesváry (1966) for a ratio of mixing length to pressure scale-height equal to 1.5. Notice that the right-hand boundary of this figure corresponds to the center of the Sun.

base of the common envelope of W UMa stars must lie at depths where $\delta D/H \ll 1$; otherwise, the energy delivered by the envelope to the different halves of the common photosphere will enjoy little horizontal coupling.

Figure 1 shows that depths H greater than only a fraction of a percent of D suffice to satisfy $\delta D/H \ll 1$. If the base of the common envelope lies at such levels or deeper, the emergent flux from the photosphere will bear no relation to the locations of the ultimate sources of heat—the two stellar interiors—because the thermal inertia of the intervening common envelope buffers us from seeing the individual luminosities. In the layers where $\delta D/H \ll 1$, a very slight amount of fluid flow suffices to redistribute the heat completely. Thus, in these layers the horizontal distribution of the diffusive heat flux is forced to be whatever is consistent with hydrostatic equilibrium in the frame which corotates with the orbital motion (we assume synchronous spins). This *vector* condition requires the pressure and density to be uniform on equipotentials; i.e., we have the barotropic condition: $P = P(\Phi)$ and $\rho = \rho(\Phi)$, with

$$dP/d\Phi = -\rho ,$$

where Φ is the effective gravitational potential referred to the corotating frame. If we further assume chemical homogeneity in the envelope, the equation of state plus Saha's equation imply that the temperature and the mean molecular weight are also uniform on equipotential surfaces: $T = T(\Phi)$ and $\mu = \mu(\Phi)$.

We shall see in § II that the barotropic condition leads to the surface layers being characterized by a nearly uniform effective temperature. When this argument is turned around, the observed smallness of color variations of W UMa stars with orbital phase implies that the parameter δ must be quite small at the base of the common envelopes of these systems. A very small value of δ is, therefore, a fundamental constraint on any viable theory of the interior structure of W UMa stars, and it is the fundamental starting point of the theory advanced by Shu, Lubow, and Anderson (1976; see also Lubow and Shu 1977).[2]

II. FLUX DISTRIBUTION IN COMMON ENVELOPES

The horizontal distribution of flux delivered to the photosphere by a barotropic envelope depends on whether the common envelope is radiative or convective. Consider first the case of contact binaries with common radiative envelopes—ignoring here the issue of whether these systems are theoretically possible from the point of view of

[2] This reasoning leads us to agree with the criticism of Hazlehurst and Meyer-Hofmeister (1973) of the model of Biermann and Thomas (1972). We differ, however, with the former authors' conclusion that contact binaries must evolve toward unit mass ratio because we do not require that the properties of the system be continuous across the inner critical surface.

stellar interiors (Lucy 1968*a*; Lubow and Shu 1977). The radiative flux in optically thick regions is given by the diffusion approximation:

$$\boldsymbol{F}_{\rm rad} = -\frac{4caT^3}{3\kappa\rho}\,\nabla T = \left(\frac{4caT^3}{3\kappa\rho}\frac{dT}{d\Phi}\right)(-\nabla\Phi)\,. \qquad (3)$$

Denote $-\nabla\Phi$ by $\boldsymbol{g}$ and notice that the coefficient of $\boldsymbol{g}$ in the above equation is a function of Φ alone. Thus, the coefficient does not vary with horizontal position on an equipotential surface. If $\boldsymbol{F}_{\rm rad}$ is ultimately radiated by the photosphere at the rate σT_e^4, we obtain the Eddington–von Zeipel law of gravity brightening (von Zeipel 1924; Eddington 1926):[3]

$$T_e \propto g^{0.25}\,, \qquad (4)$$

where g now represents the local surface gravity. To derive equation (4) we make no assumption that the photosphere lies *exactly* on an equipotential; we require only that the transition from layers where $\delta D/H \ll 1$ to layers where $\delta D/H \gg 1$ occupies a *spatially thin* region so that the gravity at the *top* of the barotropic envelope is well approximated by the surface value. In the Roche model, g is typically 10% larger in average on the side of the large star than on the side of the small star. Equation (4) taken by itself then predicts that eclipsing systems with radiative envelopes would generally appear brighter during occultation than during transit. Observed light curves which exhibit the latter behavior are called A-type (Binnendijk 1970).

Equation (3) violates radiative equilibrium, i.e., $\nabla\cdot\boldsymbol{F}_{\rm rad} \neq 0$, and fluid flow must arise to compensate for the local radiative heating and cooling of the gas (Eddington 1929; Sweet 1950). Through conservation considerations of mass and momentum, the small circulation velocities $\boldsymbol{u}$ so induced lead to small density and pressure fluctuations. Fortunately, the perturbations of ρ, κ, and T from their zero-order relations, $\rho = \rho(\Phi)$, $\kappa = \kappa(\Phi)$, and $T = T(\Phi)$, are generally small; thus, the radiant flux of photons (which we can see; we cannot see the material energy flux, $\rho\boldsymbol{u}[h + |\boldsymbol{u}|^2/2 + \Phi]$) should still be well approximated by the zero-order relation (4).

Consider, next, the case of common convective envelopes. By analogy with the Eddington–von Zeipel derivation of equation (4), we again assume that P, ρ, μ, and T are functions of Φ alone. Then, the usual mixing-length formulation (e.g., eq. [6-281] of Mihalas 1970) gives the convective heat flux in the denser regions as

$$\boldsymbol{F}_{\rm conv} = \frac{1}{4\sqrt{2}}\,\rho T\left(\frac{\partial h}{\partial T}\right)_P \left|lg\left[1 - \left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P\right]\right|^{1/2} \left|\frac{l\nabla P}{P}\left[\frac{d\ln T}{d\ln P} - \left(\frac{\partial \ln T}{\partial \ln P}\right)_s\right]\right|^{3/2}\boldsymbol{n}\,,$$

where s is the specific entropy, $\boldsymbol{n} = -\boldsymbol{g}/|\boldsymbol{g}|$ is the unit outward normal, and l is the mixing length. We adopt the conventional local hypothesis that l is α times the pressure scale-height, $P/|\nabla P|$, where α is a pure number of order unity, often taken to be 1.5. Writing $\nabla P = \rho\boldsymbol{g}$, we obtain

$$\boldsymbol{F}_{\rm conv} = \left(\frac{\alpha}{2}\right)^2 \rho T\left(\frac{\partial h}{\partial T}\right)_P \left\{\frac{P}{2\rho}\left[1 - \left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P\right]\right\}^{1/2} \left[\frac{d\ln T}{d\ln P} - \left(\frac{\partial \ln T}{\partial \ln P}\right)_s\right]^{3/2}\boldsymbol{n}\,. \qquad (5)$$

For a given choice of α, the coefficient of $\boldsymbol{n}$ in the above equation is a function of Φ alone. Thus, if the flux in the deeper layers is carried (almost) entirely by convection, then the photosphere must ultimately radiate the left-hand side of equation (5) at the rate σT_e^4. In this situation, equation (5) gives

$$T_e \propto g^{0.00}\,. \qquad (6)$$

The gravity dependence drops out altogether, and equation (6) implies that, in the absence of other effects, every element of the surface looks equally bright. If we do not admit to a breakdown of conventional mixing-length theory, the observed slight differences between occultation and transit in W UMa systems of late spectral type must arise from higher-order effects, or from photospheric effects such as limb darkening and reflection (see §§ II and III).

At this point, we should make three explanatory remarks. First, we note that Lucy (1967) has arrived at a different law of convective gravity brightening, $T_e \propto g^{0.08}$, by referring to mixing-length computations of Baker and Temesváry (1966). Lucy implicitly stacks next to one another a continuum of inward integrations appropriate to spherical dwarf stars, without regard for mechanical balance in the horizontal directions. The value of the specific entropy in the deeper layers is not likely to be inaccurately determined in this way, but its *slight gradient*, which is necessary to produce a nonzero convective flux, should not be obtained by such a procedure. In the deeper layers, P not only has a uniform functional dependence on T (Lucy's criterion), but it also has one on Φ. The latter consideration determines the horizontal distribution of the entropy gradient and gives equation (6) as the appropriate form of the gravity brightening law in common convective envelopes.

Second, we comment that we have not included the interaction between rapid rotation and convection in the derivation of equation (6). Although the precise nature of this interaction under astrophysical conditions is not

[3] The conventional terminology, "gravity darkening," is a misnomer.

well understood (Durney 1971; Spiegel 1972; Gilman 1974), there are two effects of potential importance for the theory of the light curves of contact binaries: (*a*) the action of the Coriolis force may not only reduce the overall efficiency of convection but may also lead to different efficiencies at the poles relative to the equators, i.e., effectively to values of α in equation (5) which vary with horizontal position on an equipotential surface; and (*b*) the convection pattern may become time-dependent, leading to unsteady dynamo action and starspot cycles which produce asymmetric and variable photometric light curves. Lacking an adequate theory to estimate the magnitude of these effects, we are forced to ignore them in our models of synthetic light curves (§§ III and IV).

Third, we should estimate the spectral type of W UMa stars for which we expect a transition of the distribution law, $T_e \propto g^\beta$, from $\beta = \frac{1}{4}$ to $\beta = 0$. According to Böhm-Vitense (1958), heat transport in the subphotospheric layers of dwarf stars is usually effected totally by radiation or totally by convection; mixed modes occur only for a very narrow range of spectral types. The crossover occurs roughly at spectral type A7 (Böhm-Vitense as quoted on p. 611 of Cox and Giuli 1968). However, the convection zones of dwarf stars with spectral type A7 are so thin that they are likely to constitute only the outermost layers of the common envelope. How thick does the outer convection zone have to be to change the flux distribution from the law, $F \propto g^1$, established in the lower radiative layers to the law, $F \propto g^0$, appropriate for barotropic convection zones? The answer is obvious according to our discussion of § I: the parameter $\delta D/H$ at the bottom of the convection zone must be substantially greater than unity. Figure 2 shows a plot against effective temperature (and spectral type) of $\log_{10}(\delta D/H)$ evaluated at the bottom of the convection zone of the envelope models constructed by Baker and Temesváry (1966) for main-sequence stars. From Figure 2 we deduce that the transition from the classical gravity brightening law, $\beta = \frac{1}{4}$, to the convective law, $\beta = 0$, should occur roughly in a narrow range of spectral types centered at about F5. As we shall discuss in § IV, we believe this finding to explain Ruciński's (1974) discovery that the transition from A-type behavior to W-type behavior of the light curves of W UMa stars occurs, with only a few exceptions, abruptly as a function of effective temperature. In this paper we shall not use Ruciński's model-dependent definitions of A and W types; we shall revert to Binnendijk's (1970) terminology which assigns A/W type to shallower/deeper minimum at occultation. If we then examine the systems listed by Binnendijk, we find that the transition from A-type behavior to W-type behavior does indeed occur at about spectral type F5.

III. PHOTOSPHERES OF CONTACT BINARIES

The zero-order gravity brightening laws, equations (4) and (6), give the basic information we require of the flux distribution to make model atmospheres calculations of W UMa systems. These photospheric calculations are complicated by the need to treat the "reflection effect." We believe that at least part of the W-type phenomenon, i.e., a system appearing brighter in transit than in occultation, owes its explanation to the reflection effect. Thus, it is important to give this effect a better and more physical treatment than has been customary in the literature so far.

For W UMa stars, there is a relatively simple way to give a unified treatment of both the reflection effect and limb darkening by appealing to a modified form of Eddington's approximation. The Eddington approximation is

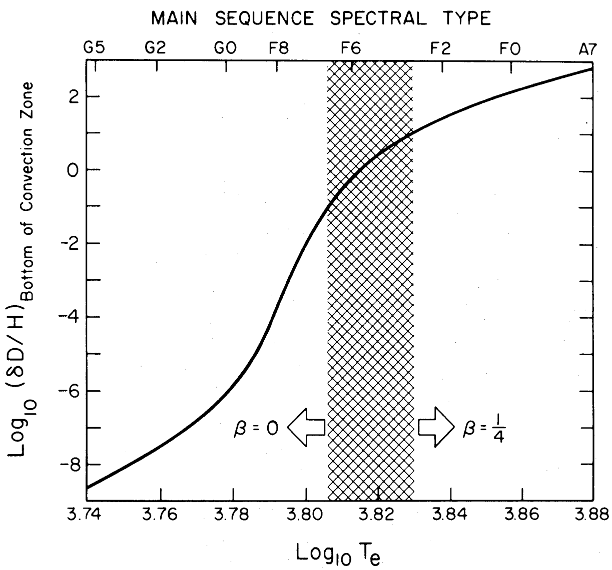

FIG. 2.—The ratio of the dynamical time scale to the thermal time scale, $\delta D/H$, evaluated at the bottom of the convection zone for main-sequence stars of different effective temperatures. This graph is based on the calculations of Baker and Temesváry (1966) for the chemical composition: $X = 0.70$, $Y = 0.27$, $Z = 0.03$, and for a ratio of mixing length to pressure scale-height equal to 1.

basically a diffusion approximation for the photospheric layers of a plane-parallel gray atmosphere. If the atmosphere is in radiative equilibrium and F is the flux that radiation is required to carry, the integration of equation (3) yields the following distribution of T^4 as a function of vertical optical depth τ:

$$\tfrac{1}{3}caT^4 = F(\tau + Q_0)\,,$$

where Q_0 is an integration constant whose value remains to be determined.

The atmospheres of very cool dwarf stars, however, will not have a linear T^4-τ relation because, except at very small optical depths, convection carries a significant fraction of the total flux (see, e.g., Auman 1969). To derive the implications of this behavior for the light curves of W UMa stars, let us suppose that $R(\tau)$ represents the fraction of the total flux F carried by radiation at optical depth τ:

$$F_{\rm rad} = FR(\tau)\,, \qquad F_{\rm conv} = F[1 - R(\tau)]\,, \tag{7}$$

with $R(0) = 1$. In a radiative atmosphere $R(\tau) = 1$ for all τ, but in a convective atmosphere $R(\tau) < 1$ for $\tau > 0$ with $R(\tau) \to 0$ as $\tau \to \infty$. The transition from the total flux being carried entirely by convection to its being carried entirely by radiation generally occurs at a very *large* value of $\delta D/H$; thus, there is no horizontal redistribution of the value of $F = \sigma T_e^4$ established from below. In other words, in systems with deep convective envelopes, there is no switch from equation (6) to the gravity brightening law (4) despite the fact that all *atmospheres* must ultimately become radiative at sufficiently small optical depths.

The substitution of the first of equations (7) into equation (3) now yields the generalized T^4-τ relation:

$$\tfrac{1}{3}caT^4 = FQ(\tau)\,, \tag{8}$$

where $Q'(\tau) = R(\tau)$. Notice that the function $Q(\tau)$ contains an arbitrary additive constant, say $Q_0 \equiv Q(0)$, in its definition as the indefinite integral of $R(\tau)$. We proceed below to evaluate Q_0.

The emergent specific intensity at $\tau = 0$ is given by

$$I(\mu) = \int_0^\infty \frac{caT^4}{4\pi} e^{-\tau/\mu} \frac{d\tau}{\mu}\,,$$

where μ is the cosine of the angle between the viewing direction and the local zenith. The substitution of equation (8) into the above relation yields, after an integration by parts,

$$I(\mu) = \frac{3F}{4\pi}[Q_0 + \mathscr{R}(\mu)]\,, \qquad \mathscr{R}(\mu) \equiv \int_0^\infty R(\tau)e^{-\tau/\mu}d\tau\,. \tag{9}$$

Notice that $\mathscr{R}(\mu)$ is related to the Laplace transform of $R(\tau)$ and that the functional form of $Q_0 + \mathscr{R}(\mu)$ governs the form of the limb darkening law. Since $R(\tau) \leq 1$, $\mathscr{R}(\mu) \leq \mu$ with equality if $\mu = 0$ or if $R(\tau) = 1$.

To compute Q_0—in particular, to calculate its variation as a function of position on the common surface of the contact binary—we suppose that the sky (i.e., the set of lines of sight which do not ultimately intercept the photosphere) occupies ω_S steradians, with $\omega_S < 2\pi$ for those locations which suffer the "reflection effect." On the average, no net energy is exchanged between two photospheric elements visible to each other if both elements have (nearly) the same effective temperature. Therefore, to maintain detailed thermal equilibrium in the photospheres of W UMa stars, the flux F which comes directly from below must ultimately be radiated entirely into the solid angle ω_S:

$$F = \int_{\omega_S} I(\mu)\mu d\omega = 3F\left(\frac{\omega_S}{4\pi}\right)[Q_0\langle\mu\rangle_S + \langle\mu\mathscr{R}(\mu)\rangle_S]\,, \tag{10a}$$

$$\langle\mu\rangle_S \equiv \frac{1}{\omega_S}\int_{\omega_S}\mu d\omega\,, \qquad \langle\mu\mathscr{R}(\mu)\rangle_S \equiv \frac{1}{\omega_S}\int_{\omega_S}\mu\mathscr{R}(\mu)d\omega\,. \tag{10b}$$

Equation (10a) can be solved to obtain the desired result,

$$Q_0 = \frac{1 - 3\langle\mu\mathscr{R}(\mu)\rangle_S S}{3\langle\mu\rangle_S S}\,, \tag{11}$$

where we have written $S = \omega_S/4\pi$ to denote the fraction of the celestial sphere which is occupied by the local sky. Equations (9) and (11), together with the variation of $F = \sigma T_e^4$ implied by equation (4) or (6), complete our derivation of the combined effects of gravity brightening, limb darkening, and reflection.

To appreciate the physical meaning of equation (11), let us consider a photosphere which is in radiative equilibrium. (Such an atmosphere may overlie either a radiative or a convective *envelope*.) If $R(\tau) = 1$ at all optical depths from which radiation can effectively emerge, equations (8), (9), and (11) become

$$T^4 = \tfrac{3}{4}T_e^4(Q_0 + \tau)\,, \qquad I(\mu) = \frac{3\sigma T_e^4}{4\pi}(Q_0 + \mu)\,, \qquad Q_0 = \frac{1 - 3\langle\mu^2\rangle_S S}{3\langle\mu\rangle_S S}\,, \tag{12}$$

where we have written $\sigma T_e^4 = caT_e^4/4$ for F. Let us further perform the averages (10b) not over the actual sky, but over a hemisphere instead. The latter unbiased averages give $\langle\mu\rangle = \frac{1}{2}$ and $\langle\mu^2\rangle = \frac{1}{3}$; with this pedagogical approximation, Q_0 becomes

$$Q_0 \approx \frac{2}{3}\left(\frac{1 - S}{S}\right).$$

Notice that Q_0 has the classical value $\frac{2}{3}$ if $S = \frac{1}{2}$ and that equations (12) then correspond to the usual Eddington approximation for a normal stellar atmosphere. As we move on the surface of a contact binary and lines of sight above the local horizon become filled with pieces of star, S becomes less than $\frac{1}{2}$, Q_0 becomes greater than $\frac{2}{3}$, and the temperature rises at each value of τ. This heating of the atmosphere results from the need to push out the same flux F through a smaller window (the sky), and the latter feat is accomplished by raising the overall radiation pressure without changing its gradient with respect to optical depth. The outer layers ($\tau \ll Q_0$) are heated noticeably by the reflection effect, but the deeper layers ($\tau \gg Q_0$) are little affected by the additional deposition of radiant energy density (as long as S does not approach zero). Thus, a simple reference to Figure 1 shows that, typically, the incident radiation is deposited at depths where fluid flow can do little to redistribute the heat; this conclusion justifies our implicit assumption that the absorbed radiation is reradiated locally. Notice also that reflection does not change our definition (cf. eqs. [4] and [6]) of the effective temperature because the net flux—including both inward and outward contributions from the reflection effect—remains equal to F, the internal value. Reflection does, however, make the atmosphere hotter and more isothermal.

The departure of the more exact equation (11) from the approximate law, $Q_0 \approx 2(1 - S)/3S$, can be understood as follows. First, in the regions where the local sky is biased to low values of μ (e.g., the regions near the "neck" of the Roche model), the atmosphere heats up more because it must push out the requisite radiation predominantly "sideways." Second, the appearance of $\mathscr{R}(\mu)$ instead of μ in equations (9) and (11) results from convective atmospheres having reduced amounts of limb darkening in comparison with radiative atmospheres. In particular, in the extreme case when $F_{\rm rad}/F \equiv R(\tau)$ is zero except for a vanishingly small range near $\tau = 0$, we infer $\mathscr{R}(\mu) \simeq 0$, and equations (9) and (11) imply

$$I(\mu) = \frac{\sigma T_e^4}{\pi}(4\langle\mu\rangle_S S)^{-1}\,. \tag{13}$$

Such a completely convective photosphere shows no limb darkening at all, but it does exhibit reflection. When $S < \frac{1}{2}$, an observer sees a higher specific intensity than usual, when $S = \frac{1}{2}$, because photons have been "reflected" from other parts of the system. It does not matter whether these additional photons have been scattered into the line of sight or first absorbed and then reemitted, since they must all come out ultimately via the sky.

More generally, we comment that one of the elegant features of equations (9) and (11) is that they guarantee the total power radiated into the celestial sphere exactly to equal the total luminosity generated in the two interiors. A similar line of reasoning in the context of detached binaries was advanced by Eddington (1926) to justify the naming of the effect as "reflection."

IV. SYNTHETIC LIGHT CURVES

a) *Procedure*

The deliberations of the previous sections may be summarized in the following formula for the emergent specific intensity:

$$I(\mu) = \frac{3\sigma\overline{T}_e^4}{4\pi}\left(\frac{g}{\bar{g}}\right)^{4\beta}[Q_0 + \mathscr{R}(\mu)]\,, \tag{14}$$

where $\beta = \frac{1}{4}$ and $\mathscr{R}(\mu) = \mu$ for systems with radiative envelopes, whereas $\beta = 0$ and $\mathscr{R}(\mu)$ is given by equation (9) for systems with convective envelopes. For practical applications we distinguish between radiative and convective *envelopes* at spectral type F5, and we ignore here the slight amount of convection which occurs in the *photospheres* of stars with spectral type between F5 and A7.

In equation (14), Q_0 is given by equation (11); $\bar{g}$ is the gravity averaged over the surface of the contact binary; and $\sigma\overline{T}_e^4$ equals the total luminosity divided by the surface area A. Equation (14) is derived from an approximate theory, but it incorporates all of the generally recognized effects in a simple and physically plausible fashion. We expect that its adoption would lead to reasonably accurate bolometric light curves for contact binaries.

Following Lucy (1968*b*), we note that monochromatic light curves can also be readily obtained by using the (modified) Eddington-Barbier approximation for gray atmospheres (see, e.g., p. 72 of Kourganoff 1963):

$$I_\nu(\mu) = B_\nu(T(\tau)) \text{ at a value of } \tau \text{ where } \int_0^\tau R(t)dt = \mathscr{R}(\mu)\,. \tag{15}$$

In equation (15), $B_\nu(T)$ is the Planck function, and $T(\tau)$ in our problem has the functional form implied by equations (8) and (9):

$$T(\tau) = \bar{T}_e\left(\frac{g}{\bar{g}}\right)^\beta \left\{\frac{3}{4}\left[Q_0 + \int_0^\tau R(t)dt\right]\right\}^{1/4}. \tag{16}$$

Notice that the value of τ in equation (15) is chosen so that the integral of equation (15) over frequency ν would recover equation (14).

In the remainder of this section, we give a few computed examples. The notation we use is as follows: mass ratio, $q = M_{II}/M_I$; filling factor, $f = (\Phi_s - \Phi_1)/(\Phi_2 - \Phi_1)$, with $f = 0$ for marginally filled common envelopes and $f = 1$ for totally filled common envelopes; inclination angle i, with $i = 0°$ corresponding to pole-on viewing and $i = 90°$ corresponding to equator-on viewing; and orbital phase ϕ, with $\phi = 0$ corresponding to transit (primary minimum for A types, secondary minimum for W types) and $\phi = 0.5$ corresponding to occultation. In the above definition of f; Φ_s, Φ_1, and Φ_2 are, respectively, the values of the equipotentials (in the Roche model) at the stellar surface, the inner critical surface, and the outer critical surface.

If we are dealing with systems of very late spectral type, it is further necessary to specify the function $R(\tau)$ by consulting detailed model atmospheres calculations (e.g., Auman 1969). It is then necessary to know $\bar{g}$ and $\bar{T}_e$; hence, this procedure is best left to the modeling of individual systems. In this paper, we wish to survey some general trends; thus, we shall use only the two opposite extremes of limb darkening laws: $\mathscr{R}(\mu) = \mu$ and $\mathscr{R}(\mu) = 0$ (cf. eqs. [12] and [13]). Moreover, we shall normalize all light curves to unit intensity at maximum light (elongation). Then, for the bolometric light curves, we need to specify only the gravity brightening exponent β, the mass ratio q, and the filling factor f; for the monochromatic light curves, we also need to specify the average effective temperature $\bar{T}_e$.

Our computational method is to divide the stellar surface into 3072 star area elements, distributed with fourfold symmetry, and the celestial sphere into 1646 sky solid-angle elements of roughly 25 square degrees each. From each star element, we calculate the quantities, $\langle\mu\rangle_S S$ and $\langle\mu^2\rangle_S S$ associated with the fraction S of sky elements which have an unobscured line of sight to the given star element. The intensity contribution of the star element to these sky elements is then computed, with appropriate geometrical weights, and added to each sky element bin. This process is repeated for each star element until each sky element has received all of the light from the system to which it is entitled. The light curves at different inclinations i as a function of orbital phase are then found by examining (and normalizing) the appropriate ring of sky elements. By making use of the fourfold symmetry of the Roche model, we reduce the necessary computational steps by a factor of 4 over the simplified description given above. A check on our final program was performed by choosing $\mathscr{R}(\mu) = \mu$, $q = 0.5$, $f = 0.37$, $\beta = 0.08$; ignoring the reflection effect; examining the $i = 80°$ ring of sky elements; and comparing the resulting bolometric light curve to that published as Figure 3 of Lucy (1968*b*). This check gave agreement to the accuracy of the line drawing.

b) *Computed Examples*

Figure 3 shows the set of bolometric light curves appropriate to $q = 0.5$ and $f = 0.5$; in Figure 3*a*, $\beta = \frac{1}{4}$, $\mathscr{R}(\mu) = \mu$ (radiative photosphere overlying a common radiative envelope); in Figure 3*b*, $\beta = 0$, $\mathscr{R}(\mu) = \mu$ (radiative photosphere overlying a common convective envelope); and in Figure 3*c*, $\beta = 0$, $\mathscr{R}(\mu) = 0$ (completely convective photosphere overlying a common convective envelope). As was anticipated in § II, Figure 3*a* shows that contact binaries with common radiative envelopes generally exhibit A-type light curves, with the occultation minimum being shallower than the transit minimum. A detailed investigation reveals, however, that for inclination angles i between 90° and 75°, gravity brightening per se and reflection favor W-types, and that it is the darkening effect of having two sets of limbs at transit versus one set at occultation which manages to keep the system darker at transit than at occultation. For i between 75° and 38°, reflection does not offset the combined effects of gravity brightening and limb darkening, and the light curves do not become W-type until after the inclination angle becomes less than 38°. These findings are consistent with Ruciński's discovery that W UMa stars of early spectral type almost always show A-type light curves.[4]

The differences between the curves of Figures 3*a* and 3*b* arise solely from the absence of the effects of gravity brightening in Figure 3*b*. Between $i = 90°$ and $i = 70°$ the light curves of Figure 3*b* show A-type behavior because

[4] Exceptions to this general rule are likely to be found only for systems of spectral type middle F. The flagrant exception SV Cen (Ruciński 1976; Wilson and Starr 1976) is apparently not in thermal equilibrium and may have accretion luminosity. For this accretion luminosity to manifest itself in appreciably different surface temperatures for the two stellar components, it is necessary that the common envelope be tenuous enough so that its base does not satisfy the requirement $\delta D/H \ll 1$.

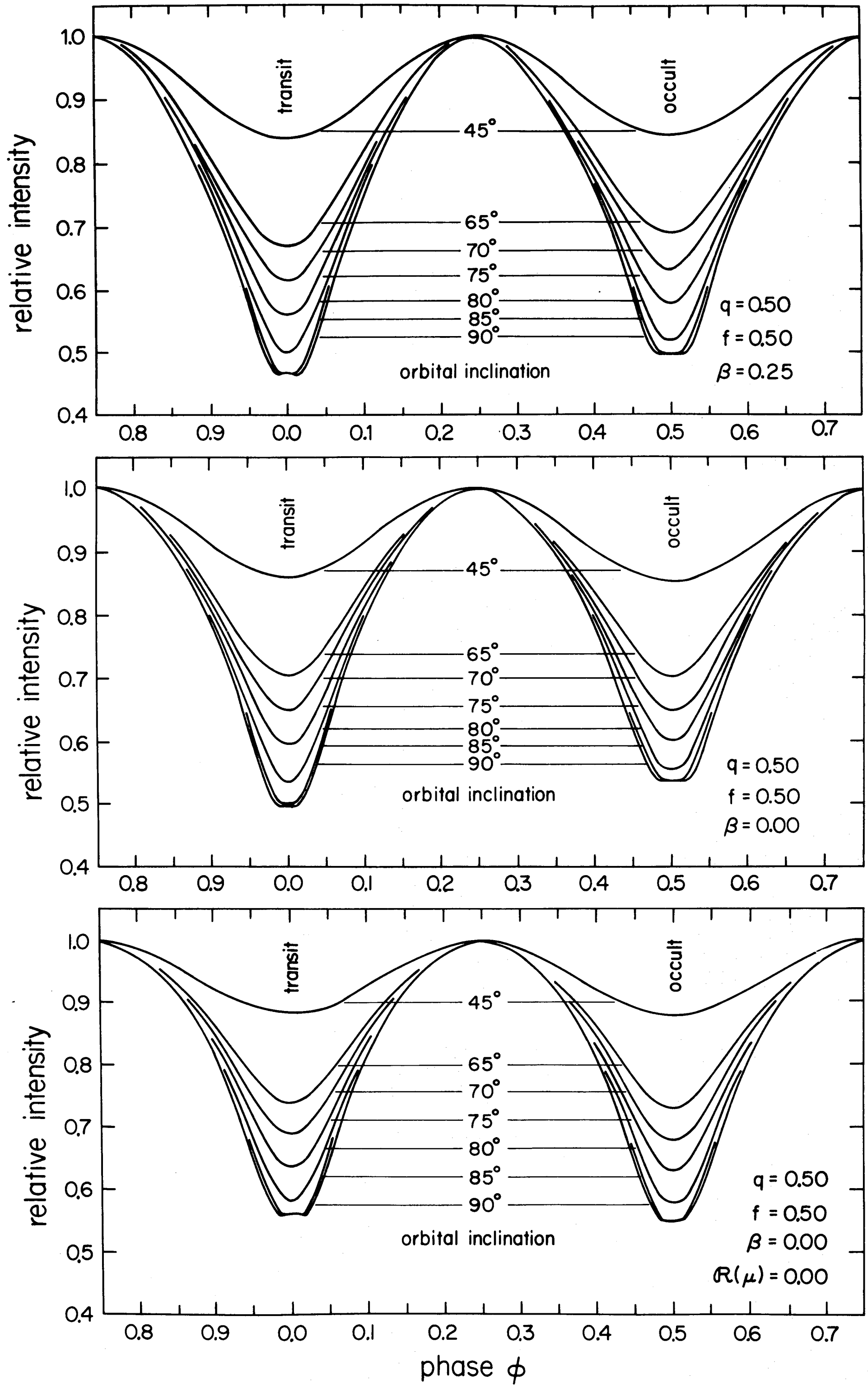

FIG. 3.—Theoretical bolometric light curves for contact binaries with mass ratio $q = 0.5$ and filling factor $f = 0.5$, viewed at different orbital inclinations. (*a*) (*top*) A system with a common *radiative* envelope: at inclinations $i > 38°$, the light curves are A-type, showing a deeper transit minimum. (*b*) (*middle*) A system with a common *convective* envelope, calculated with *full* limb darkening: at inclinations $i < 70°$, the light curves are W-type, showing a deeper occultation minimum. (*c*) (*bottom*) A system with a common *convective* envelope, calculated with *zero* limb darkening: at all inclinations, the light curves are W-type.

the darkening effect of having two sets of limbs outweighs the brightening effect of a small amount of reflection at transit. As i decreases, however, more of the limb of the large star is covered up at transit and more of the reflection brightened region near the neck is revealed. Thus, for i less than about 70° we have a transition to W-type light curves. The exact value of the transition inclination changes somewhat with varying q and f, but it remains typically between 70° and 75° when $\mathscr{R}(\mu) = \mu$.

The differences between Figures 3*b* and 3*c* arise solely from the absence of the effects of limb darkening in Figure 3*c*. As one would anticipate, Figure 3*c* shows W-type light curves at all inclination angles from $i = 90°$ to $i = 0°$ because of the presence of reflection. Since Figures 3*b* and 3*c* represent opposite extremes of possible limb darkening laws, we may conclude that W UMa stars of spectral type later than F5 will show A-type light curves only when they are viewed nearly equator-on. Even then, W-type light curves will be seen if limb darkening is sufficiently reduced by photospheric convection. In any case, in W UMa stars of late spectral type we expect to see W-type light curves exlusively if the systems are inclined by less than about 70°.

A tendency does exist for W UMa systems of spectral type later than F5 to exhibit W-type light curves if they suffer partial eclipses, and to exhibit A/W type light curves for radiative/convective photospheres if they suffer total eclipses. Consider the 15 systems of spectral type later than F5 which are common to Table 5 of Binnendijk (1970) and Tables 2 and 3 of Ruciński (1974). Of these, 10 systems show partial eclipses, and all 10 are W-type. (Ron Webbink [private communication] points out, however, that selection effects may play a role since the A-types have smaller mass ratios and, thus, may be selected for total eclipses.) Of the five systems which show total eclipse, the two A types (AK Her and FG Hya) are of spectral type F8 and G0 and can be expected to have radiative photospheres, whereas two of the W types (RZ Com and AH Vir) are of spectral type K0 and may have reduced limb darkening laws because of appreciable photospheric convection. The remaining completely eclipsing W type, W UMa itself, is of spectral type F8 and is the only system which apparently violates our general rule.

There is another aspect about Figures 3*b* and 3*c* which we should mention. Even for Figure 3*c*, the most W-like light curves occur at $i = 90°$ and $i = 75°$, and have occultation minima which are deeper than the transit minima only by about 0.02 mag. The last number can be roughly doubled for monochromatic light curves at the mean wavelength of the broad-band blue filter when $\bar{T}_e = 5000$ K because of the great sensitivity of the Planck function to slight temperature variations (which arise from the heating of the reflection effect, for example) at frequencies where $h\nu/kT \gg 1$. In contrast, for observed W-type light curves the occultation minima are typically deeper than the transit minima by about 0.09 mag in blue light, with the spread ranging from about 0.02 mag to 0.2 mag.

It is possible to obtain blue light curves of slightly more extreme W types in our models by lowering the average effective temperature $\bar{T}_e$, or by decreasing the filling factor f, or by decreasing the mass ratio q somewhat from $q = 0.5$. To illustrate these effects, the upper solid curve in Figure 4 compares the ratio of blue light intensities at transit and at occultation when $q = 0.25$, $f = 0.1$, $\beta = 0$, $\mathscr{R}(\mu) = \mu$, $\bar{T}_e = 4500$ K. We see that the maximum W-type behavior corresponds to a magnitude difference of 0.04 at i between 65° and 60°. The lower solid curve shows the depth of the primary minimum to be rather shallow at this inclination, amounting to only 0.29 mag (intensity ratio of minimum/maximum = 0.77). The dashed curves in Figure 4 show the same quantities, except now $\mathscr{R}(\mu)$ has been set equal to zero. The maximum magnitude difference between the two minima is only increased from 0.04 to 0.05 by the absence of limb darkening, although it now occurs when the primary minimum is still respectably deep. We expect that the most important change effected by a more realistic reduction of limb darkening because of convection in the photosphere will be a shift of the transition angle of A-type to W-type behavior toward larger inclinations. In any case, it is clear that our theory would have to press to a rather unlikely corner of parameter space to produce W-type light curves with magnitude differences in B of as much as 0.1 between the two minima.

V. DISCUSSION

In this paper we have constructed a physical theory for the light curves of W UMa stars based on the concept that the dynamical time scale is very short in comparison with the thermal time scale at the base of the common envelope. There are two improvements of our approach over previous treatments: (*a*) standard mixing length theory is used to derive the flux distribution in common convective envelopes in a manner which parallels the Eddington-von Zeipel theory for radiative envelopes; (*b*) a better derivation is given for the reflection effect which utilizes the special property of nearly uniform effective temperatures over the surface of a contact binary. As a bonus, our efforts lead us to a promising explanation for the distinction between A-type and W-type light curves without resorting to artificial "temperature excesses."

In our model, the difference between A and W types arises primarily because of the difference between radiative and convective envelopes, a result anticipated already by the important work of Ruciński (1974). Our contribution is to give the theoretical basis for expecting the transition in A/W behavior to occur at about spectral type F5. In addition, we have shown that equilibrium systems with common convective envelopes can show A-type behavior if we view them nearly equator-on. An observational correlation of A/W type with total/partial eclipse in systems of late spectral type exists, but is weak. We call for more observations.

On the theoretical side, our models are incomplete in that they neglect the interaction between rapid rotation

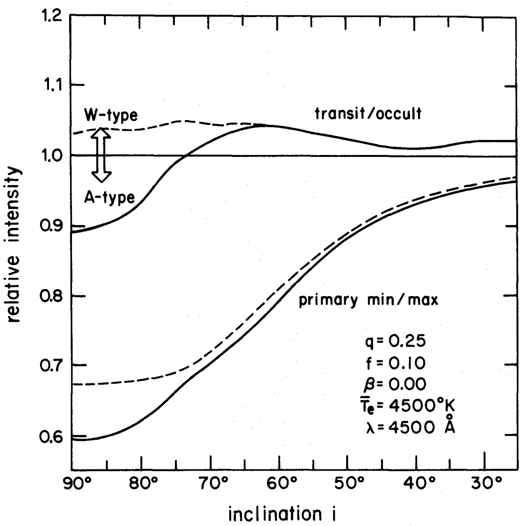

FIG. 4.—*Upper curves*, the ratio of minimum intensities, transit/occultation, as a function of orbital inclination for a contact binary with mass ratio $q = 0.25$, filling factor $f = 0.10$, common *convective* envelope, average effective temperature $\bar{T}_e = 4500$ K, and viewed at $\lambda = 4500$ Å. *Lower curves*, the ratio of the deeper minimum to the maximum intensity. The solid curves are calculated with *full* limb darkening; the dashed, with *zero* limb darkening.

and convection. On the observational side, our models suffer in their failure to produce as extreme W-type light curves as are sometimes observed. These two deficiencies may be related.[5]

It is a modern article of faith that the coupling between rotation and convection lies at the heart of stellar dynamo action (Parker 1970). Starspot activity undoubtedly accompanies the regeneration of surface magnetic fields by a migratory dynamo, and this activity may well explain the asymmetries and variabilities associated with the light curves of W UMa stars of late spectral type. Perhaps such effects, along with pole-to-equator variations of the convective efficiency, also contribute on the level of a few percent to the depths of the minima in the more extreme W types (for a more drastic view, see Mullan 1975).

There are many lines of evidence which support the notion of enhanced magnetic activity induced by the interaction between convection and faster than normal stellar rotation. (In detached binaries, see Kron 1952; in W UMa stars, see, e.g., Kuhi 1964 and Binnendijk 1970.) It is comparatively easy to accept the proposition that starspots might cause asymmetries and variabilities of the light curves; after all, how could one prevent the effect? Such blemishes on celestial bodies are probably an unavoidable aspect of the real world (Galileo 1610). However, why should they contribute to the extremeness of the W-type phenomenon in many observed systems? For this effect to arise, spot activity must occur preferentially on the more massive star. We offer the following speculation.

Parker (1975) has suggested that the growth of the strength of the surface magnetic fields in stars with convective envelopes may be limited by magnetic buoyancy. If all other factors are equal, stars with deeper convective envelopes would be able to generate and to hold onto stronger magnetic fields. In standard models of main-sequence dwarfs, a smaller star would have a deeper convective envelope. However, in the models of W UMa stars with common convective envelopes proposed by Shu, Lubow, and Anderson (1976), the convective envelope of the small star is divided into two zones by a stable contact discontinuity at the inner critical surface of the (modified) Roche model, whereas the convective envelope of the large star continues uninterrupted past its Roche lobe. Indeed, in the detailed calculations of Lubow and Shu (1977), the convective envelope of the large star is deeper than normal. Hence, we find it plausible that the magnetic fields which emerge on the common surface on the side of the large star are stronger than those which emerge on the side of the small star.

If our speculation on this point is correct, it would mean that we must wait for fundamental advances in theories of convection and dynamo action before we will be able to model actual light curves much beyond the current level of precision. The effects of starspots on the light curves are unlikely, however, to be very large; hence, the theory developed in this paper should be sufficiently accurate to be used by observers to obtain reasonable estimates of the fundamental system parameters of W UMa stars. In this regard, it should be noted that the synthetic light curves displayed in Figures 3*a*–3*c* differ appreciably from one another for the *same* values of q, f, and i. Obviously, the light curves of contact binaries are somewhat sensitive to the details of the laws adopted for gravity brightening,

[5] It cannot be argued that extreme W types arise because the assumption $\delta \ll H/D$ is violated for systems with extremely thin common envelopes. If δ becomes appreciable at the base of the common envelope, we should be able to see *better* the underlying individual stars. This uncovering of the individual sources of light and heat would strongly favor A-type light curves; indeed, it may even make the minima so unequal as to wipe out the characteristic feature which distinguishes EW (W UMa-like) light curves from EB (β Lyrae–like) light curves.

limb darkening, and reflection. To help observers reduce their data, therefore, we plan to produce a theoretical atlas of bolometric light curves *and* idealized velocity profiles which spans the extreme cases (radiative envelopes with radiative photospheres, convective envelopes with completely radiative photospheres, and convective envelopes with completely convective photospheres) and which covers all of parameter space (q, f, i) with a reasonable mesh. Judicious interpolation should make for more accurate applications to real systems. This atlas may also be useful as a stepping stone to further refinements of the theory or more detailed studies of individual systems. Details concerning our atlas will be published elsewhere.

We thank Steve Lubow and B. Paczyński for some very stimulating discussions. We are grateful to S. Ruciński for sending us various helpful preprints and reprints; we have benefited greatly from reading his comprehensive accounts of W UMa stars. The numerical calculations for this work were performed at the Berkeley Computing Center. This project is supported in part by NSF grants AST 75-02181 and AST 75-04809.

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