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ON THE STRUCTURE OF CONTACT BINARIES. I. THE CONTACT DISCONTINUITY

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ABSTRACT

The problem of the interior structure of contact binaries is reviewed, and a simple resolution of the difficulties which plague the theory is suggested. We propose that contact binaries contain a contact discontinuity between the lower surface of the common envelope and the Roche lobe of the cooler star. This discontinuity is maintained against thermal diffusion by fluid flow, and the transition layer is thin to the extent that the dynamical time scale is short in comparison to the thermal time scale. The idealization that the transition layer has infinitesimal thickness allows a simple formulation of the structure equations which are closed by appropriate jump conditions across the discontinuity. The further imposition of the standard boundary conditions suffices to define a unique model for the system once the chemical composition, the masses of the two stars, and the orbital separation are specified. We leave the actual construction of stellar models for a later communication.

Subject headings: stars: binaries — stars: interiors

I. INTRODUCTION

This paper is the first of a series which deals with the structure and evolution of contact binaries. Binary stars which are in physical contact present intriguing problems for the theory of stellar interiors and atmospheres. They are known to exist as pairs of main-sequence stars in W Ursae Majoris systems (see the review by Hazlehurst 1975), and they are believed to arise almost inevitably in the Case A mass transfer in semidetached binaries (see the reviews by Paczynski 1971 and Plavec 1973).

No completely successful theory exists for the structure of contact binaries, although Osaki (1965) and Lucy (1967, 1968*a*, *b*) have made the important suggestion that a crucial role must be played by a common envelope which redistributes and radiates away the sum of the luminosities generated in the two separate cores. This beautifully simple idea explains the striking observed property that the two components of W UMa systems have nearly the same effective temperatures despite their having quite different masses. Although some difficulties still remain (Rucinski 1974), there can be little doubt that the gross properties of the *photospheres* of W UMa systems receive an adequate explanation on the basis that the two stars are in physical contact and share a common envelope.

The structure of the *interiors* of contact binaries remains a puzzle. This paper reviews the difficulties with the present theories and proposes a simple resolution for the existing impasse. The further development of this work by Lubow and Shu (1967*a*, *b*, hereinafter referred to as Papers II and III) aims at the actual construction of zero-age main-sequence models (Paper II), and at the dynamical problem of mass and angular momentum loss when evolutionary processes have led the system to come into contact with its outer critical surface (Kuiper 1941, Paper III). The eventual goal is to be able to compute the full evolutionary history of contact binary systems—a program for which no one has yet suggested a credible procedure. This capability is of crucial importance for investigating problems such as Kraft's (1965) suggestion that W UMa systems are the progenitors of novae and dwarf novae.

II. DIFFICULTIES WITH THE PRESENT THEORETICAL STATUS

a) Kuiper's Paradox

The challenge posed for an adequate theory of the stellar interiors of contact binaries is exhibited most forcefully by recalling the difficulties of constructing contact systems composed even of two main-sequence stars. We refer to the set of circumstances which gives rise to this difficulty as "Kuiper's paradox."

Kuiper (1941) showed that two stars cannot be in physical contact unless their common outer surface falls in between the inner critical surface and the outer critical surface of the Roche model for close binary stars.[1] (See Fig. 1.) The space between the two critical surfaces is fairly thin in comparison with the interiors; thus, Kuiper argued that the average radii of the two stars, $R_{\rm I}$ and $R_{\rm II}$, are effectively constrained to the approximate Roche-lobe ratio,

$$(R_{\rm I}/R_{\rm II})_{\rm Roche\,lobe} = (M_{\rm I}/M_{\rm II})^{0.46} . \quad (1)$$

[1] The validity of this conclusion requires the two stars to spin synchronously with respect to the motion of the circular orbit. Empirically, this requirement is met by W UMa stars which show absorption lines whose widths are compatible with the assumption of synchronism.

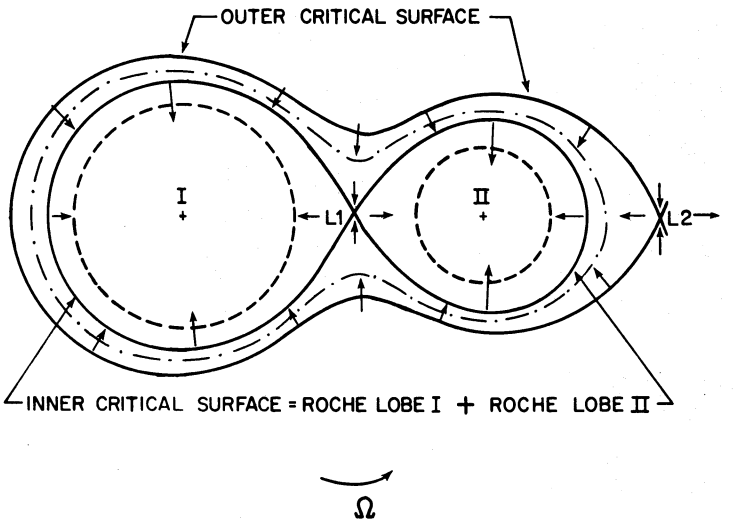

FIG. 1.—The inner and outer critical surfaces of the Roche model plotted in the equatorial plane. The arrows are perpendicular to the equipotentials and indicate the direction of the effective gravitational field. The effective gravity vanishes at the inner and outer Lagrangian points, L_1 and L_2. Gravitational confinement of a gas against the expansive tendency of its internal pressure is possible only if the physical surface(s) of the system lies beneath the outer critical surface. For a detached binary, the two stellar surfaces (*dashed curves*) both lie beneath the inner critical surface; for a contact binary, the common stellar surface (*dashed-dotted curve*) lies between the inner and outer critical surfaces.

Such a mass-radius relation is generally incompatible with the physical requirements of two ordinary stars. For example, the mass-radius relation derived from studies of single stars shows

$$(R_{\rm I}/R_{\rm II})_{\rm main\ sequence} = (M_{\rm I}/M_{\rm II})^n , \qquad (2)$$

where the exponent n is approximately 0.6 on the upper main sequence and has a value close to 1 on the lower main sequence. Clearly, equations (1) and (2) are incompatible unless $M_{\rm I}$ and $M_{\rm II}$ are exactly equal. This is Kuiper's paradox. Kuiper concluded that two stars cannot be in stable contact unless they have unit mass ratio; however, we now know that W UMa systems show typical mass ratios on the order of 2:1 (Binnendijk 1970).

b) Lucy's Model

Lucy (1968*a*) presented a mathematical proof which seemed to support Kuiper's conclusion if the common envelope is radiative, but not if it is convective. Lucy's proof depends on a continuity of physical properties at the inner critical surface, and we shall argue in § IV that the breakdown of this assumption provides precisely the key to the solution of Kuiper's paradox.

Nevertheless, it is an observed fact that many of the W UMa systems have late enough spectral types (F–K) to justify Lucy's hypothesis that their common envelopes are convective. For these systems, we would support Lucy's elegant proposal that the specific entropy of the gas, s, in the convective region common to the two stars must have a single numerical value and that this value is adjusted so that the shared surface can radiate the sum of the luminosities generated in the two separate cores.

To see this, suppose that the specific entropy were to develop significant *horizontal* variations as some workers have imagined. (The word "vertical" in this paper refers to the local direction of the effective gravity.) These variations would be erased on a *dynamical* time scale and not a thermal time scale as has previously been supposed. The reason, of course, is that the equation of hydrostatic equilibrium requires the pressure P and the density ρ to be constant on a closed equipotential surface; and in a chemically homogeneous medium, this condition requires the specific entropy s to have a uniform value also. If the specific entropy were to develop significant horizontal variations, the resulting pressure differences would cause gas to slide dynamically along equipotentials until these variations were erased. If, in addition, the convective transport of heat is efficient, then s cannot show significant vertical variations either. Therefore, s must have a single value in the convective region common to both stars.

Unfortunately, Lucy implicitly assumed the additional constraints, $s_{\rm I} = s_{\rm II} = s$, where $s_{\rm I}$ and $s_{\rm II}$ are the specific entropies in the separate convection zones beneath the inner critical surface, and s is the entropy in the common envelope. This restriction led to untenable constraints on the modes of energy generation in the stellar cores of Lucy's models. Lucy found the artificial result that he could not construct zero-age systems unless one of the stars burned hydrogen via the proton chain and the other, via the CNO cycle. Moreover, the total mass of his model

sequence A spanned only a small range in the neighborhood of 2.5 $M_\odot$, in direct contrast with the much larger observed range.

To overcome these problems, various works (Hazlehurst 1970; Moss and Whelan 1970; Biermann and Thomas 1973; Flannery 1976; Lucy 1976) have attempted to improve upon the comparisons with observations by dropping one or another of Lucy's original assumptions: (1) the specific entropy of the common envelope has a single value, (2) the component stars are zero-age main-sequence stars, and (3) the system is in a quasi-static state of equilibrium. Unfortunately, any significant relaxation of one of these assumptions introduces difficulties of its own without retaining the simplicity of Lucy's original picture. (See the review by Hazlehurst 1975.)

We do not propose to discard any one of Lucy's basic assumptions for W UMa stars (other than to allow the common envelope to have either a convective or a radiative structure). We propose to introduce another degree of freedom: the freedom of steady circulatory flow (see also the comment by Lynden-Bell 1975), and the resultant establishment of a contact discontinuity. We present the physical picture in § III, and we give the mathematical formulation in § IV. In Paper II, we show that this degree of freedom allows us to construct unique zero-age models of two stars with arbitrarily chosen masses placed at a distance apart appropriate for any desired degree of contact. Our method also allows us to construct static contact binaries with evolved cores, but we have not formulated the equivalent of a Henyey technique to follow rapid phases of stellar evolution.

III. THE CONTACT DISCONTINUITY MODEL

To motivate the following discussion, we first notice that the conflict between equations (1) and (2) arises because of a competition between the mechanical requirements of the system and the thermal requirements. Fluid flow must arise to resolve the conflict introduced by the uneven distribution of the sources of heat. Notice, however, that material can flow extremely slowly insofar as considerations of mechanical equilibrium are concerned, and still redistribute heat effectively. The inequality between the competition is reflected in the dynamical time scale, $t_{\rm dyn}$, in dwarf stars and subgiants being many orders of magnitude shorter than the thermal (Kelvin-Helmholtz) time scale, $t_{\rm K\text{-}H}$.[2]

In contact binaries, horizontal interchange of mass and heat occurs effectively only in the part of the envelope which lies above the inner critical surface. Only in this part of the envelope are the closed equipotential surfaces common to both stars (see Fig. 1), and only in this region can the gas easily slide from one star to the other because of slight horizontal differences in the pressure.[3]

Below the inner critical surface (the Roche lobes), however, horizontal redistribution between the two stars is much more difficult because access to the other star without doing work against gravity is available only at the inner Lagrangian point, L_1. Inside the Roche lobes, therefore, the two stars are nearly decoupled, and we must allow for the possibility of a nearly discontinuous behavior across the inner critical surface.

The strong mechanical requirements imply that the pressure must be continuous across the inner critical surface, but they allow the temperature and the density and/or their gradients to be discontinuous (i.e., we consider the possibility of contact and/or weak discontinuities). However, cold gas cannot sit stably on top of hot gas because rising elements of hot gas and descending elements of cold gas would destroy such a contact discontinuity on a dynamical time scale. It would also be physically unrealistic to allow the gas above the critical surface to be hotter than *both* stars beneath it. Thus, mechanical considerations allow only *one* star to have a contact discontinuity—the star which is colder immediately beneath its Roche lobe than immediately above it. The other star will show continuous behavior across its Roche lobe; at best it can exhibit a weak discontinuity (continuity of the variables P, ρ, and T but not of their derivatives—see § IV).

How is the contact discontinuity maintained against thermal diffusion? Consider, first, the case when both stars have efficient convective envelopes, and we can ignore the radiative transport (Fig. 2*a*). If $s_{\rm II} < s_{\rm I}$, then star II exhibits the contact discontinuity. Rising convective elements interior to the Roche lobe of star II cannot penetrate into the common convective envelope (where the specific entropy s equals $s_{\rm I}$), because the resulting buoyancy deficit opposes such penetration. Star II behaves as a saucepan of boiling water which has a lid placed over it. How does the heat generated at the bottom get out? By boiling over the lip!

A slight heating of the interior of star II under constant volume generates a slight excess of pressure which pushes the gas over the lip at L_1 into the interior of star I. Once there, it is nearly at the same pressure as the ambient medium, but it is considerably colder and denser. Thus, it drops into the deep interior of star I where it is eventually heated up to become $s_{\rm I}$ material. This process, if it occurs irregularly, may cause small erratic changes in the period; but, more importantly, it displaces preexisting $s_{\rm I}$ material and causes it to rise above the Roche lobe of star I and to flow into the common envelope. The gas above the Roche lobe of star II is not heated from below, but it has the structure of a convection zone because it is *mechanically* coupled to the convection zone in star I.

[2] The local ratio $t_{\rm dyn}/t_{\rm K\text{-}H}$ is roughly $F/\rho ha$, where F is the diffusive heat flux, ρh is the enthalpy per unit volume, and a is the sound speed. Consider a contact binary whose common envelope fills one-half of the volume between the inner and outer critical surfaces. The base of this common envelope would lie below the photosphere by about 10% of the size of the individual stars. At a corresponding depth in the Sun, the ratio $F/\rho ha \approx 10^{-7}$.

[3] The "astrostrophic" tendency for the horizontal flow to occur nearly parallel to isobars presents a complication. However, this dynamical effect would not prevent the common envelope from mixing thoroughly on a thermal time scale.

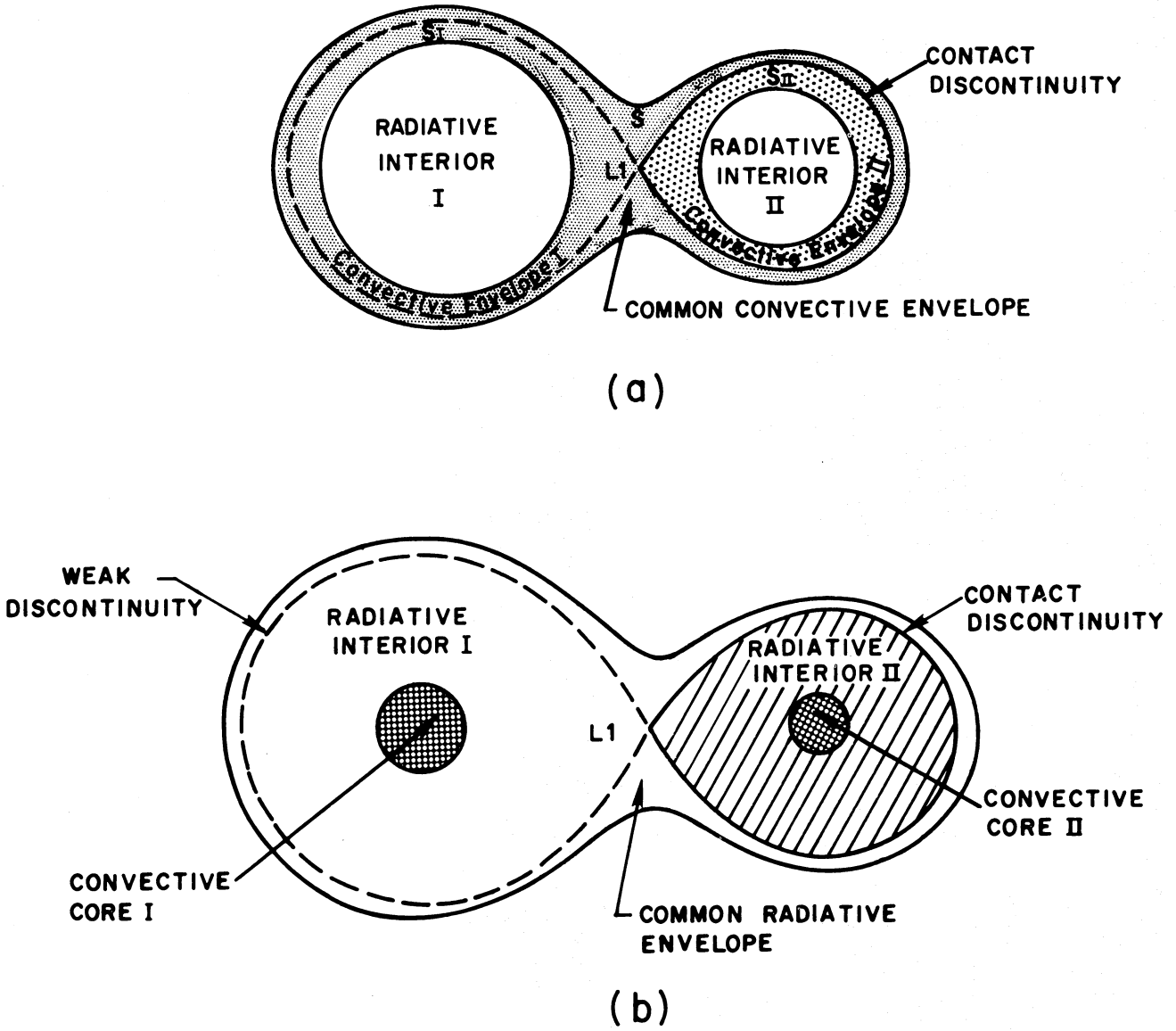

FIG. 2.—Schematic diagram of the structure of contact binaries. (*a*) A low-mass contact binary at zero age with a common convective envelope. The specific entropy s_{II} beneath the Roche lobe of star II is less than the specific entropy s_I beneath the Roche lobe of star I. For dynamical stability, the specific entropy s in the common envelope equals s_I. (*b*) A high-mass contact binary at zero age with a common radiative envelope. The temperature T_{II} immediately beneath the Roche lobe of star II is less than the temperature T_I immediately beneath the Roche lobe of star I. For dynamical stability, the temperature immediately above the inner critical surface equals T_I.

The gas above the Roche lobe of star II slowly loses heat. Besides the stellar surface, these losses are especially severe in the transition layer which constitutes the actual contact "discontinuity." Inside the transition layer at the Roche lobe of star II, diffusive effects (such as convective overshoot or radiative transfer) plus circulation currents slowly convert s_I material into s_{II} material. In a steady state, this conversion exactly balances the loss of s_{II} material through the L_1 region.[4] This, then, is our picture of how a contact discontinuity is maintained in the case of two convective stellar envelopes. A crude order-of-magnitude argument shows the transition layer over star II to be thin to the extent that $(t_{dyn}/t_{K\text{-}H})^{2/3}$ is locally small.

Consider, next, the more complicated case when both stars have radiative envelopes (Fig. 2*b*). If T_{II} below the Roche lobe of star II is less than T_I below the Roche lobe of star I, star II exhibits the contact discontinuity. Moreover, if we ignore the contributions of fluid flow to the heat transport, star I would have a weak discontinuity.

[4] This assertion is the weakest part of our argument. If the balance is not exact, there may be a secular shift of the mass ratio. Perhaps the predominance of the mass ratio 2:1 among observed systems owes its existence to such secular shifts (see also § IV*d*).

To see this, notice that the temperature just above the inner critical surface equals T_I, and the temperature gradient implies a radiative flux which upon integration over the entire critical surface is just sufficient to carry the sum of the luminosities generated in the two cores. On the other hand, just below the Roche lobe of the two stars, the temperature gradients are those required to carry the individual luminosities. This requirement would result in a discontinuity in the normal derivative of T at the Roche lobe of star I.

In practice, any discontinuity in the derivative of T in star I would be smoothed by fluid flow over a finite region of space *even in the limit* $t_{\rm dyn}/t_{\rm K\text{-}H} \to 0$. As we shall see, the actual redistribution of the heat generated in the two cores takes place in a region of fluid flow *beneath* the Roche lobe of star I; we refer to this region as the "mixing region." To simplify the discussion in § IV of the present paper, we shall not try to model the details of this heat redistribution by a complicated and uncertain flow calculation; instead, we shall ignore the finite size of the actual mixing region and simulate its net effect by adopting a weak discontinuity at the Roche lobe of star I.

Imagine, now, cool material near L_1 on the side of star II to be initially in pressure equilibrium with corresponding hot material near L_1 on the side of star I and with hot gas in the common envelope. Radiation flows to the cool gas from below as well as from the hot gas, causing the pressure to rise slightly. Since the restraint of gravity is absent at L_1, the resultant pressure imbalance must drive cold material from star II to flow over the lip at L_1 into the interior of star I. Once inside star I, this material is no longer buoyant in the ambient medium, and it drops into the interior of star I as before. The overflow rids star II of the heat generated in its interior: this process also displaces gas in the interior of star I into the common radiative envelope. In a steady state, the high-temperature material which flows on top of star II would again be converted into low-temperature material over the entire transition layer at the Roche lobe of star II, with the thickness of this layer being adjusted so that the rate of this diffusive conversion exactly compensates for the rate of loss of gas through L_1. The entire flow cycle provides the mixing process mentioned earlier.

The scenarios for any combination of convective and radiative envelopes are equally easy to work out in this qualitative fashion. To calculate the flow rigorously and quantitatively would, however, be an involved task. Fortunately, for the problem of stellar structure, the existence of the very small parameter $t_{\rm dyn}/t_{\rm K\text{-}H}$ greatly simplifies the practical problem, and we can avoid computing the details of the flow altogether. If we are not interested in the structure of the transition layer, we can ignore both the diffusive effects and the slow flow which maintain this structure, and we can simply approximate the entire transition as a contact discontinuity. Similarly (but with less physical justification), to simplify the heat redistribution problem in stars with radiative envelopes, we can model the mixing region as a weak discontinuity. *The "jump conditions" for the discontinuities together with the idealized stellar structure equations, which apply for the regions away from the discontinuities, suffice to give a complete mathematical specification of the system when the boundary conditions are given* (§ IV).

It may sound paradoxical that we can ignore the very effect—i.e., fluid flow—that maintains the contact discontinuity, but analogous results occur in other fields. For example, the photoionization cross section and the collisional cross section play similar roles in the theories of Strömgren spheres and shock waves (Spitzer 1968; Landau and Lifshitz 1959). The crucial point is the following. Whatever difficulties the diffusive mechanisms encounter in carrying the required heat flux can be compensated by a very slight amount of fluid flow because $t_{\rm dyn}/t_{\rm K\text{-}H}$ is so small. In a steady state, then, the fluid flow easily accomplishes whatever is required of it. It almost does not matter what the flow is, as long as it exists!

To be sure, much interesting physics occurs inside the mixing region for the redistribution of the separate luminosities generated in the cores. In a future investigation, we hope to return to examine the approximation involved in our replacement of the net effects of the mixing region by a weak discontinuity. (We remark here that this uncertainty does not affect convective regions since the heat transport problem in stellar interiors is then conventionally replaced by the approximation that the superadiabatic temperature gradient is negligible.) In our first efforts to construct stellar models, we have adopted this simplification for the mixing process in radiative envelopes because we wish to emphasize that the critical concept for the resolution of Kuiper's paradox lies in the *contact* discontinuity, and not in the detailed mechanism for the redistribution of heat. The concept of the contact discontinuity is secure as long as the ratio of $t_{\rm dyn}/t_{\rm K\text{-}H}$ is locally small.

IV. THE STELLAR STRUCTURE EQUATIONS FOR CONTACT BINARIES

a) Formulation of the Basic Equations

In this section, we give our formulation of the mathematical problem for the stellar interiors of contact binaries. We make the *a priori* assumption that, apart from slow circulation currents, the system rotates as a rigid body with angular frequency Ω, which remains to be determined (see eq. [5]). In the corotating frame, then, the effective potential Φ is given by

$$\Phi = \mathscr{V} - \tfrac{1}{2}\Omega^2\varpi^2 , \tag{3}$$

where ϖ is the axial radius from the center of mass of the system, and $\mathscr{V}$ is the true gravitational potential. The latter satisfies Poisson's equation

$$\nabla^2\mathscr{V} = 4\pi G\rho . \tag{4}$$

For an arbitrary density distribution ρ, satisfying hydrostatic equilibrium, Ω can be obtained from the scalar virial theorem (Chandrasekhar 1961). Thus, for the present problem,

$$\Omega = -\left(\frac{1}{2}\int \mathscr{V}\rho dV + 3\int PdV\right)\Big/ J, \tag{5}$$

where the integrations are carried out over the volume of the entire mass distribution, and where J is the total angular momentum of the system,

$$J = I\Omega, \qquad I = \int \varpi^2 \rho dV. \tag{6}$$

Notice that for a contact binary there is no clear distinction between orbital and spin angular momentum; only the total would be conserved in any evolutionary process not involving mass loss and/or gravitational radiation.

We refer to equations (3)–(6) as the dynamical equations of the system because they relate to the dynamics of the rotation and the geometry of the gravitational field. To complete our description of the system, we need the analogs of the usual stellar structure equations. As is well known, there is considerable advantage in expressing these equations in the corotating frame and in using the effective potential as the relevant coordinate variable (see, e.g., Faulkner, Roxburgh, and Strittmatter 1968; Kippenhahn and Thomas 1970; and, especially, Mark 1968). For our problem, the most useful form for the level-surface structure equations is the following set:

$$\frac{dP}{d\Phi} = -\rho, \tag{7}$$

$$-\frac{4acT^3}{3\kappa\rho}\frac{dT}{d\Phi} = \frac{L_{\rm rad}(\Phi)}{\bar{g}A(\Phi)}, \tag{8a}$$

$$L_{\rm rad} = L \quad \text{(radiative regions)}, \tag{8b}$$

$$\frac{ds}{d\Phi} = 0 \quad \text{(efficient convective regions)}, \tag{8c}$$

$$\frac{dL}{d\Phi} = \rho\epsilon\overline{g^{-1}}A(\Phi), \tag{9}$$

$$P = \frac{\rho}{\mu m_{\rm H}}kT + \tfrac{1}{3}aT^4, \tag{10a}$$

$$\kappa = \kappa(\rho, T, \text{chem. comp}), \qquad s = s(\rho, T, \text{chem. comp.}), \qquad \epsilon = \epsilon(\rho, T, \text{chem. comp.}). \tag{10b}$$

Mark (1968) was the first to give the structure equations in essentially the above form for systems where an effective potential can be usefully defined. We provide an alternative derivation in the Appendix which makes explicit the nature of the approximation involved for application to contact binaries, and which justifies the use of these equations even in the presence of thermally important hydrodynamical flow when the equations are supplemented by appropriate jump conditions.

In equations (7)–(10), P, ρ, T, κ, s, ϵ, and μ have their usual meanings except they have been averaged over the equipotential surface, Φ = constant; L is the total energy (excluding neutrinos) which crosses the equipotential surface per unit time; $L_{\rm rad}$ is the radiative luminosity; $A(\Phi)$ is the area of the closed equipotential surface; and $\bar{g}$ and $\overline{g^{-1}}$ are the values of the effective gravity and the reciprocal of the effective gravity averaged over the equipotential surface:

$$\bar{g}A(\Phi) \equiv \int_{\Phi=\text{const.}} |\nabla\Phi| dA, \qquad \overline{g^{-1}}A(\Phi) \equiv \int_{\Phi=\text{const.}} \frac{dA}{|\nabla\Phi|}. \tag{11}$$

It is possible to use the divergence theorem and equations (3) and (4) to derive the well-known relationship,

$$\bar{g}A(\Phi) = 4\pi GM(\Phi) - 2\Omega^2 V(\Phi),$$

where $M(\Phi)$ and $V(\Phi)$ are the mass and volume interior to the equipotential Φ. These latter quantities can be obtained by integrating equations (A13) of the Appendix.

Equations (7)–(10) are exact relations to the extent that circulation currents are dynamically unimportant (because the flow speed is very subsonic) and carry no *net* heat flux across a closed equipotential surface. The latter assumption must be violated in the mixing region, so $L_{\rm rad} \neq L$ inside this region even if it is formally stable to convection according to Schwarzschild's criterion. In general, then, neither equation (8b) nor equation (8c) can

describe the mode of energy transport in the mixing region or in the transition layer. In contrast, equations (7) and (9) are valid even inside the mixing region and the transition layer (as long as L is defined to be the total energy carried by radiation, convection, and circulation currents); thus, the jump conditions for P and L can be derived by integrating these equations across the discontinuities.

b) Jump Conditions

We suppose that the exact Φ has a geometrical structure similar to the Roche potential with the following features: (1) an inner critical surface containing two lobes ("Roche lobes") that join at a saddle-point (L_1), and (2) an outer critical surface passing through a second saddle-point (L_2). We denote the value of Φ at the inner critical surface by Φ_1, and at the outer critical surface by Φ_2. The "centers" of the two components are defined by the two extreme values of Φ where $\bar{g}A$ and $\overline{g^{-1}}A$ vanish; let us call these extreme values $\Phi_{c\mathrm{I}}$ and $\Phi_{c\mathrm{II}}$. For $\Phi_2 > \Phi > \Phi_1$, the equipotential surfaces enclose both centers, and $\bar{g}A$ and $\overline{g^{-1}}A$ are single-valued functions of Φ. For $\Phi_1 > \Phi > \max(\Phi_{c\mathrm{I}}, \Phi_{c\mathrm{II}})$, the functions $\bar{g}A$ and $\overline{g^{-1}}A$ are double-valued; we denote the separate branches by the subscripts I and II depending on the lobe interior. To emphasize in what follows the two separate lobes which make up the inner critical surface, we shall denote the value of the equipotential at the Roche lobes by Φ_{RL} instead of Φ_1; numerically, of course, $\Phi_{\mathrm{RL}} = \Phi_1 = \Phi_{\mathrm{L1}}$.

From the definitions (11), we easily drive the jump conditions for the functions $\bar{g}A$ and $\overline{g^{-1}}A$:

$$\bar{g}A(\Phi_1{+}) = \bar{g}_{\mathrm{I}}A_{\mathrm{I}}(\Phi_{\mathrm{RL}}{-}) + \bar{g}_{\mathrm{II}}A_{\mathrm{II}}(\Phi_{\mathrm{RL}}{-})\,, \tag{12a}$$

$$\overline{g^{-1}}A(\Phi_1{+}) = \overline{g_{\mathrm{I}}^{-1}}A_{\mathrm{I}}(\Phi_{\mathrm{RL}}{-}) + \overline{g_{\mathrm{II}}^{-1}}A_{\mathrm{II}}(\Phi_{\mathrm{RL}}{-})\,. \tag{12b}$$

The jump conditions on equations (7) and (10) are easily obtained by integration across the Roche lobes under the assumption that the right-hand sides remain bounded:

$$P(\Phi_1{+}) = P_{\mathrm{I}}(\Phi_{\mathrm{RL}}{-}) = P_{\mathrm{II}}(\Phi_{\mathrm{RL}}{-})\,, \tag{13a}$$

$$L(\Phi_1{+}) = L_{\mathrm{I}}(\Phi_{\mathrm{RL}}{-}) + L_{\mathrm{II}}(\Phi_{\mathrm{RL}}{-})\,. \tag{13b}$$

From our previous discussion, we realize the jump condition on the temperature must take the form

$$T(\Phi_1{+}) = \text{larger of } T_{\mathrm{I}}(\Phi_{\mathrm{RL}}{-}) \quad \text{and} \quad T_{\mathrm{II}}(\Phi_{\mathrm{RL}}{-})\,. \tag{14}$$

The equivalent jump conditions for convective envelopes can also be expressed as

$$s(\Phi_1{+}) = \text{larger of } s_{\mathrm{I}}(\Phi_{\mathrm{RL}}{-}) \quad \text{and} \quad s_{\mathrm{II}}(\Phi_{\mathrm{RL}}{-})\,. \tag{15}$$

Equations (12)–(15) complete the necessary specification of the jump conditions across the contact discontinuity.

For the following discussion, let us assume $T_{\mathrm{II}}(\Phi_{\mathrm{RL}}{-}) < T_{\mathrm{I}}(\Phi_{\mathrm{RL}}{-})$ so that the contact discontinuity occurs at Roche lobe II. If the common envelope is radiative and if we ignore the contribution of the fluid flow to L by setting $L = L_{\mathrm{rad}}$, we require a weak discontinuity at Roche lobe I. (As we cross Roche lobe I, the right-hand sides of eqs. [8a] and [8b] vary discontinuously in accordance with eqs. [12a] and [13b], whereas P and T—and therefore ρ and κ—vary continuously. Thus, $dT/d\Phi$ given through eq. [8a] must vary discontinuously.) This discontinuous behavior of the radiative flux is required to preserve the total luminosity L. In reality, fluid flow in the mixing region allows L_{rad} to differ from L which would presumably result in a continuous variation of $dT/d\Phi$.

We now notice that, for a given chemical composition, the ordinary differential equations (7)–(10) with the associated jump conditions (12)–(14) form a complete set to solve for P, ρ, T, and L if we know the functional dependence of $\bar{g}A$ and $\overline{g^{-1}}A$ on Φ. This latter information may be obtained from equations (11) once we know the spatial dependence of $\Phi(x, y, z)$. The direct solution of $\Phi(x, y, z)$ from the partial differential equation (4) would couple the structure equations to the dynamical equations and would result in a very difficult technical task. Physical intuition suggests, however, that it should be possible to approximate an adequate solution of the dynamical equations by a slight modification of the standard Roche model.

c) The Modified Roche Model

The key to our proposed technique rests on the following two facts: (1) For the stellar structure problem, we only need the function $\Phi(x, y, z)$ to compute the coefficients (11). (2) To high accuracy, the true $\Phi(x, y, z)$ will not differ very much from the effective potential associated with two *unperturbed* spherical stars of radii and masses R_{I}, R_{II}, and M_{I}, M_{II}, placed at an appropriate distance D apart.

The functional form of the effective potential Φ_* associated with the latter configuration, written in an obvious notation, is easily computed as

$$\Phi_* = -\frac{GM_{\rm I}}{R_{\rm I}} - \int_{r_{\rm I}}^{R_{\rm I}} \frac{GM_{\rm I}(r_{\rm I}')}{r_{\rm I}'^2}\, dr_{\rm I}' - \frac{GM_{\rm II}}{R_{\rm II}} - \int_{r_{\rm II}}^{R_{\rm II}} \frac{GM_{\rm II}(r_{\rm II}')}{r_{\rm II}'^2}\, dr_{\rm II}' - \tfrac{1}{2}\Omega^2\varpi^2\,, \tag{16a}$$

$$M_{\rm I}(r_{\rm I}) \equiv \int_0^{r_{\rm I}} 4\pi r_{\rm I}'^2 \rho_{\rm I}(r_{\rm I}')dr_{\rm I}'\,, \qquad M_{\rm II}(r_{\rm II}) = \int_0^{r_{\rm II}} 4\pi r_{\rm II}'^2 \rho_{\rm II}(r_{\rm II}')dr_{\rm II}'\,. \tag{16b}$$

Moreover, the self-terms in the numerator of equation (5) cancel for the two unperturbed spherical stars by construction, leaving only the gravitational interaction energy between the two stars. If we ignore, now, the spin moment of inertia in comparison with the orbital part because of the great central concentration of most stars, we obtain for equation (5), the usual Keplerian formula:

$$\Omega^2 = G(M_{\rm I} + M_{\rm II})/D^3 \tag{17}$$

where D is the orbital separation. In equation (16), $\rho_{\rm I}(r_{\rm I})$ and $\rho_{\rm II}(r_{\rm II})$ are assumed to be zero for $r_{\rm I} > R_{\rm I}$ and $r_{\rm II} > R_{\rm II}$, but we do not require in approximating Φ by Φ_* that the two hypothetical spherical stars cannot overlap (i.e., we do not require $R_{\rm I} + R_{\rm II} < D$). Notice that for positions outside both unperturbed stars, Φ_* is just the usual Roche potential, but inside, Φ_* achieves less negative values. Moreover, for $R_{\rm I}$ and $R_{\rm II} < D$, in addition to the usual five "Lagrangian points," $\nabla\Phi_*$ is also zero at the two centers, $r_{\rm I} = 0$ and $r_{\rm II} = 0$. (This property is a further justification for the choice [17].) For these reasons, we call Φ_* the "modified Roche Potential."

In the approximation that we can replace Φ by Φ_*, the analytical formulae (16) and (17) give a complete specification of the geometry of the gravitational field. In particular, by choosing the two unperturbed stars we wish to put together at a distance D apart, we know *a priori* the shapes and locations of the inner and outer critical surfaces associated with $\Phi_*(x, y, z)$.

Once we have approximated Φ by Φ_*, we can calculate and tabulate $\bar{g}A$ and $\overline{g^{-1}}A$ as functions of Φ. We now need to integrate the three ordinary differential equations (7), (8a) or (8c), and (9) in three different regions; this leaves nine integration constants to be determined. Four of these constants are supplied by the jump conditions (13a), (13b), and (14) or (15). Two others are given by $L = 0$ at the two centers, $\Phi = \Phi_{c\rm I}$ and $\Phi = \Phi_{c\rm II}$. Two more are determined so that the masses of the two stars have the desired values (see the next subsection for a further discussion of this point). The last constant is to be adjusted so that we can satisfy the boundary condition at the common surface, $\Phi = \Phi_s$, the value of Φ_s being defined by the equipotential where $P = 0$ is reached: (1) For a common radiative envelope, the "radiative zero" boundary condition would be $T \to 0$ as $P \to 0$. (2) For a common convective envelope, the value of the specific entropy s is fixed by a mixing-length calculation which gives the correct transport of heat in an atmosphere with average surface gravity $\bar{g}(\Phi_s)$ and average effective temperature $\bar{T}_{\rm eff} \equiv [(L_{\rm I} + L_{\rm II})/A(\Phi_s)\sigma]^{1/4}$. Efficient iterative procedures based on the Newton-Raphson method can be devised to solve the structure problem outlined above (see Paper II).

One final check needs to be mentioned. After having successfully integrated equations (7)–(10), together with the correct jump conditions (13) and (14), we need to check to make sure that the outer surface is reached on an equipotential surface which lies between the inner critical surface and the outer critical surface. If this condition is not satisfied, we can move the orbit separation D appropriately to produce a contact system of any desired degree of contact. Thus, in the approximation where we replace Φ by Φ_*, the ultimate free parameters at our disposal (for a given chemical composition) are the masses of the two stars, $M_{\rm I}$ and $M_{\rm II}$, and the separation D of their centers.

d) Exact Solution

It is instructive to devise an exact iterative method of solution to the full set of equations (3)–(10). We start with the initial guess $\Phi = \Phi_*$, $\Omega^2 = G(M_{\rm I} + M_{\rm II})/D^3$. We then solve the structure equations (7)–(10) to obtain, among other things, ρ and P as functions of Φ. With the known functional form of $\Phi = \Phi_*(x, y, z)$, we can map $\rho(\Phi)$ and $P(\Phi)$ into $\rho(x, y, z)$ and $P(x, y, z)$. We now compute the position of the center of mass for this distribution of mass. A new potential $\mathscr{V}(x, y, z)$ is computed by solving equation (4), and a new Ω is computed via equation (5). Equation (3) now gives a new iterate for $\Phi(x, y, z)$, and we can now repeat the solution of the structure equations (7)–(10) to obtain new iterates for ρ and P. This procedure can be repeated until satisfactory convergence has been established. Mathematically, this procedure is very similar to the self-consistent field method devised by Ostriker and Mark (1968) to solve the problem of the structure of rapidly rotating single stars. Here, we do not have axial symmetry, but we have made the simplifying assumption of uniform rotation which ensures that the aspherical distortions of the interiors will be small. Thus, in practice, we expect no serious error to be involved numerically in stopping at the order of approximation indicated in the previous subsection.

However, the formulation of the exact problem forces us to confront the following issue. In an approximate solution, it suffices to think of a contact binary as two stars for which we can arbitrarily specify the individual masses $M_{\rm I}$ and $M_{\rm II}$, and the orbit separation, D (or the period $2\pi/\Omega$). But the exact equations (3)–(10) make no reference to two separate stars; indeed, a little reflection shows that it is impossible to make an unambiguous separation of the mass of the common envelope into two parts. To be sure, for unevolving models this ambiguity is relatively unimportant since the common envelope generally holds a small fraction of the total mass of the system. At a certain stage of the evolution of contact binaries, however, we believe that substantial mass and angular momentum loss can occur through the outer Lagrangian point, L_2 (Paper III). To anticipate these developments, it pays to think of a contact binary as a single object, which happens to have two centers of mass concentration.

Thus, from a physical point of view (but not the most convenient, observationally), it would make better sense to specify, not $M_{\rm I}$, $M_{\rm II}$, and D, but only the total mass, $M_{\rm I} + M_{\rm II}$, the total angular momentum, J, and one other quantity. This quantity must reflect, in some sense, the bimodal distribution of mass; thus, for example, it should suffice to specify either the moment of inertia I or the period $2\pi/\Omega$. Stated differently, we claim that contact configurations with given chemical composition, total mass, and total angular momentum form a one-parameter continuum of possible solutions. The parameter which fixes the location of a given system on the sequence of possible configurations can be taken, for definiteness, to be the moment of inertia I. Different members of the sequence will generally have different total energies (and different "mass ratios"), and it might be argued that some of these equilibrium states may be secularly unstable (see n. 4). If, however, the instability proceeds on a time scale which is long in comparison with the nuclear (or gravitational radiation) time scale, the instability would be of only minor consequence for the physical problem. In this case, the "initial" state of the system would be fixed by the process responsible for the formation of the contact configuration. For zero-age contact binaries, the process is probably dynamical fission during the pre–main-sequence evolution (see, e.g., Ostriker 1970). On the other hand, evolved systems may reach a contact configuration after first passing through a semidetached state (see, e.g., Benson 1970). The further discussion of these points is beyond the scope of the present paper.

V. SUMMARY

We summarize here the principal results of the present paper. We have shown that "Kuiper's paradox" for the theory of the stellar interiors of contact binaries arises as a result of the competition between the mechanical requirements and the thermal requirements of the system. We give explicit recognition to the fact that the dynamical time scale, $t_{\rm dyn}$, in contact binaries is generally much smaller than the thermal time scale, $t_{\rm K\text{-}H}$. This allows us to resolve "Kuiper's paradox" by introducing a contact discontinuity between the common envelope and the interior of the Roche lobe of the cooler star. The corollary of this statement is that had we *not* introduced the degree of freedom associated with the contact discontinuity, we would not have been able generally to satisfy both the idealized stellar structure equations *and* their associated boundary conditions.

Our proposed solution has the virtue that it does not require a fundamental distinction between convective and radiative envelopes, nor does it require special combinations of the modes of energy generation in the stellar cores (cf. Lucy 1968*a*). We have also formulated the stellar structure problem in a form which is equally well adapted to systems with evolved cores as to chemically homogeneous systems.

The approximate treatment given in this paper should result in an accurate description of the interior's problem as long as the ratio $t_{\rm dyn}/t_{\rm K\text{-}H}$ is locally small. This assumption breaks down for the outermost layers of the system, which may have observable consequences for the photospheric properties. In particular, we speculate that the observed asymmetries in the light curves of W UMa binaries (Rucinski 1974) may be explicable in terms of the heat flux carried by the currents discussed in § III. It may be possible to devise a perturbational calculation to evaluate the importance of the hydrodynamical flow; nevertheless, it is probably safe to assume that the *main* photospheric properties of our models will be dominated by the usual considerations accorded to the existence of common envelopes (see, e.g., Osaki 1965; Lucy 1968*b*). This assumption means that our "lowest-order" models for the atmospheres would introduce few new features beyond those which are now routinely incorporated in synthetic models of the photospheres of W UMa stars (e.g., Wilson and Devinney 1971; Hutchings and Hill 1973). Thus, we can easily anticipate that these models would have light curves with the desirable property of showing little variation of the effective temperature with changing orbital phase.

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APPENDIX

The level structure equations (7), (8a), and (9) are derived from the fluid equations governing the steady-state flow in contact binaries. In a frame of reference which rotates with the angular velocity $\boldsymbol{\Omega}$ of the figure, the equation of continuity reads

$$\nabla\cdot(\rho\boldsymbol{u}) = 0\,. \tag{A1}$$

The momentum equation, written in vector-invariant form for an inviscid fluid, is

$$\rho[\nabla(\tfrac{1}{2}|\boldsymbol{u}|^2) + (2\boldsymbol{\Omega} + \nabla\times\boldsymbol{u})\times\boldsymbol{u}] = -\nabla P - \rho\nabla\Phi\,; \tag{A2}$$

whereas in the diffusion approximation, the equation of radiative transfer becomes

$$\boldsymbol{F}_{\rm rad} = -\frac{4caT^3}{3\kappa\rho}\nabla T\,. \tag{A3}$$

Finally, the heat equation reads

$$\rho T\boldsymbol{u}\cdot\nabla s = -\nabla\cdot(\boldsymbol{F}_{\rm conv} + \boldsymbol{F}_{\rm rad}) + \rho\epsilon\,. \tag{A4}$$

We use the thermodynamic relation $Tds = dh - \rho^{-1}dP$, where h is the specific enthalpy, to write equation (A4) in the more convenient form

$$\rho\boldsymbol{u}\cdot\nabla h - \boldsymbol{u}\cdot\nabla P = -\nabla\cdot(\boldsymbol{F}_{\rm conv} + \boldsymbol{F}_{\rm rad}) + \rho\epsilon\,. \tag{A5}$$

We now take the dot product of equation (A2) with $\boldsymbol{u}$ to eliminate $\boldsymbol{u}\cdot\nabla P$ from equation (A5); then equation (A1) can be invoked to put the energy equation in the form

$$\nabla\cdot\boldsymbol{F} = \rho\epsilon\,, \tag{A6}$$

where $\boldsymbol{F}$ is the total energy flux

$$\boldsymbol{F} = \rho\boldsymbol{u}(\tfrac{1}{2}|\boldsymbol{u}|^2 + h + \Phi) + \boldsymbol{F}_{\rm conv} + \boldsymbol{F}_{\rm rad}\,. \tag{A7}$$

In the interior of the system, the magnitude of the circulation velocity, $\boldsymbol{u}$, can be estimated from equation (A4) to be a small fraction of $D/t_{\rm K\text{-}H}$, the fraction f being proportional to the tidal and rotational distortion. Hence, the largest inertial term on the left-hand side of equation (A2) is $O(f/\Omega t_{\rm K\text{-}H})$ smaller in comparison with the individual terms on the right-hand side. To this order of approximation, we have hydrostatic equilibrium, $\nabla P = -\rho\nabla\Phi$, with the associated implication that $P = P(\Phi)$ and $\rho = \rho(\Phi)$. Thus, we easily derive equation (7) of the text.

If we further assume that the chemical composition is uniform on an equipotential surface, the equation of state implies $T = T(\Phi)$. Equation (A3) then yields the familiar law of gravity "darkening,"

$$\boldsymbol{F}_{\rm rad} = -\frac{4caT^3}{3\kappa\rho}\frac{dT}{d\Phi}\nabla\Phi\,, \tag{A8}$$

where the coefficient of $\nabla\Phi$ is a function of Φ only. The integral of $\boldsymbol{n}\cdot\boldsymbol{F}_{\rm rad}$, where $\boldsymbol{n} = \nabla\Phi/|\nabla\Phi|$ is the outward unit normal, over an equipotential surface is defined to be $L_{\rm rad}$. Thus, upon integration of equation (A8), we obtain equation (8a) of the text:

$$\int_{\Phi=\text{const.}}\boldsymbol{F}_{\rm rad}\cdot\boldsymbol{n}dA \equiv L_{\rm rad} = -\frac{4caT^3}{3\kappa\rho}\frac{dT}{d\Phi}\int_{\Phi=\text{const.}}|\nabla\Phi|dA\,. \tag{A9}$$

To derive equation (9) of the text, we introduce the orthogonal curvilinear coordinates (ξ, η, ζ) with $\xi = \Phi$ and with η and ζ defining the horizontal position on a closed equipotential surface (see, e.g., Kopal 1972). Let $h_\xi = |\nabla\xi|^{-1}$, $h_\eta = |\nabla\eta|^{-1}$, $h_\zeta = |\nabla\zeta|^{-1}$ denote the metric coefficients; then equation (A6) can be written as

$$\frac{\partial}{\partial\xi}(h_\eta h_\zeta F_\xi) + \frac{\partial}{\partial\eta}(h_\zeta h_\xi F_\eta) + \frac{\partial}{\partial\zeta}(h_\xi h_\eta F_\zeta) = h_\xi h_\eta h_\zeta\rho\epsilon\,. \tag{A10}$$

We now define L to be

$$L \equiv \iint_{\xi=\text{const.}} F_\xi h_\eta h_\zeta d\eta d\zeta \equiv \int_{\Phi=\text{const.}}\boldsymbol{F}\cdot\boldsymbol{n}dA\,, \tag{A11}$$

and we integrate equation (A10) over η and ζ. With the knowledge that $\rho\epsilon$ depends only on the variable ξ in the lowest order of approximation, we obtain

$$\frac{dL}{d\xi} = \rho\epsilon \iint_{\xi=\text{const.}} h_\xi h_\eta h_\zeta d\eta d\zeta \equiv \rho\epsilon \int_{\Phi=\text{const.}} h_\xi dA \,. \tag{A12}$$

Equation (A12) is equivalent to equation (9) of the text since $\xi = \Phi$, and $h_\xi = |\nabla\Phi|^{-1}$. Similarly, we easily derive the relations

$$\frac{dV}{d\Phi} = \overline{g^{-1}A} \,, \qquad \frac{dM}{d\Phi} = \rho\overline{g^{-1}A} \,, \tag{A13}$$

where $V(\Phi)$ and $M(\Phi)$ are the volume and mass interior to Φ. This completes our derivation of the level structure equations.

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