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The Parker Instability in Differentially-rotating Disks

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Summary. We investigate Parker's instability for a differentially rotating system comprised of thermal gas, magnetic field, and cosmic-ray particles. The rotation axis coincides with the direction of the vertical gravity, and the rotation is modelled to occur with linear shear. The general initial-value problem is formulated, and the condition for normal modes is obtained from this formulation. A dispersion relation is obtained for the limiting case when the growth rate (or frequency) of the wave is large in comparison with the kinematic shear rate. This dispersion relation suffices to show, in the absence of dissipation, that no finite amount of shear and rotation can ever completely stabilize Parker's mode although the growth rate of certain Fourier components can be materially reduced. Eigenvalues and eigenfunctions are computed analytically for a number of limiting cases of interest, and a few numerical examples are given. The effect of rotation is to suppress waves which are very long in the horizontal directions; the full effects of shear are more difficult to assess – numerical methods are suggested for future work. Subsidiary issues examined in this study are (i) the derivation of an alternative equation, valid in the non-linear regime and for arbitrary geometries, for the usual fluid equation adopted to describe the behavior of the cosmic-ray pressure, (ii) the distinction between environments under which Parker's mode of instability is likely to lead to convection, to cosmic-ray inflation, or to gas drainage downward to form dense clumps of matter, (iii) an explanation of the physical reasonableness of normal mode solutions with finite energy densities at infinity and the relation of such solutions to the initial-value problem.

Key words: interstellar medium – instability – magnetic field – cosmic-rays – differential rotation

I. Introduction

Parker has pointed out in a series of papers beginning in 1966 that an important mode of instability exists for an interstellar medium comprised of thermal gas, magnetic field, and cosmic-ray particles if this system is confined to a thin layer by the vertical gravity of the rest of the Galaxy. (See Parker, 1970, and earlier references therein.)

Parker has likened the instability to a magnetic Rayleigh-Taylor instability because the weight of a heavy fluid (the thermal gas) is used to hold down two light "fluids" (the magnetic field and the cosmic-ray gas), and the natural tendency of the system is to "overturn". The constraints imposed by field freezing and the purely magnetic-coupling between the thermal gas and the cosmic-ray gas introduce certain unique features not encountered in the more usual types of Rayleigh-Taylor instability.

Several different geometries have been considered (Parker, 1966, 1967b; Lerche, 1967a) but instability is found in each case. The propensity for instability and the difficulty of constructing a final equilibrium state (see, e.g., Lerche, 1967b, c; cf. Mouschovias, 1974) has led Parker to envisage a complex dynamical state for the system of interstellar gas, cosmic-rays and magnetic field (Parker, 1966, 1967a, b, 1969).

The motivation for this study was provided by the question, Why is the differential rotation of the Galaxy not a strong stabilizing influence?[1]) A naive answer might be, because the relevant length scale for Parker's mode is much too short for the rotation of the Galaxy to constitute an important modifying influence. However, this answer is not correct because the unstable wavelengths are characterized by a scale of order $2\pi H \sim 1$ kpc, where H is the vertical scale height, and the motions associated with Galactic differential rotation (either the local rotation or the shear) over such scales are more than comparable with the Alfvén, acoustic, or "turbulent" speeds associated with the general interstellar medium.

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[1]) Lerche (1967a) has also considered a problem with rotation, but he chose the rotation axis to coincide with the direction of the magnetic field. We consider the more natural problem where the rotation axis coincides with the vertical direction, and we also allow the rotation to take place non-uniformly.

It might be argued, quite correctly, that the meaningful length scale of any wavy disturbance is not the wavelength but the inverse of the wavenumber, which is a factor 2π smaller. This reduces the rotational effects accordingly. However, it is also known from work on local gravitational instabilities that the critical wavenumber associated with Jeans' instability in a non-rotating but self-gravitating layer of matter is again H^{-1} (Ledoux, 1951), and that in this case, the inclusion of the effects of rotation (essentially the Coriolis force), considered along with the effects of dispersive motions, proves sufficient to stabilize all wavenumbers providing a certain criterion is satisfied (Toomre, 1964; Goldreich and Lynden-Bell, 1965a).

To be sure, the Jeans' instability, which relies on the self-gravity of the matter principally in the directions parallel to the Galactic plane, and Parker's instability, which relies on the gravity in the vertical direction supplied by an "external agent" (namely, the disk stars), operate on wholly different physical principles. Nevertheless since motions parallel to the Galactic plane also play an important part in Parker's instability, nothing short of a detailed investigation can rule out Galactic differential rotation as a potentially important stabilizing influence.

Our results show that while differential rotation may materially reduce the growth rate of certain Fourier components of a general disturbance, it cannot completely prevent the onset of instability (see § V). Perhaps the most important implication of our analysis is that the presence of strong shear in the flow field may place a severe requirement on the spatial geometry of initial disturbances which will trigger the formation of dense clumps of interstellar gas by the action of Parker's instability.

The neglect of the effects of Galactic differential rotation until now, we believe, can be traced to a mistaken notion that Parker's instability may account for the formation of individual interstellar clouds whose dimensions and spacings are characterized by tens of parsecs or more, but hardly 1 kpc. We shall show this notion to be misleading because the environment necessary to produce instabilities on a very short scale, say, 1–100 pc, *in the direction parallel to the magnetic field*, either require unrealistically large ratios of magnetic to gas pressures, or require very weak magnetic fields resulting in *convective* motions rather than the conventional picture of gas drainage into dense pockets. We shall reserve the terminology "magnetic Rayleigh-Taylor instability" exclusively for the latter type of behavior, and we shall refer to the general class of instabilities (including the convective and cosmic-ray instabilities) as the "Parker instability". In our view, then, *cloud complexes and giant* OB-*associations*, are the natural product of the magnetic Rayleigh-Taylor instability, not individual interstellar clouds (see the paper following this one).

In carrying out the formulation for this problem and in completing an analytical study of the normal modes of the system for certain limiting cases of interest, we discovered two subsidiary points which seemed to lack sufficient elucidation in the existing literature. These are

(a) an alternative equation, valid in the nonlinear regime and for arbitrary geometries, for the usual fluid equation adopted to describe the behavior of the cosmic-ray pressure,

(b) an explanation of the physical reasonableness of normal-mode solutions with finite energy densities at infinity and the relation of such solutions to the initial-value problem.

With these preliminary remarks in mind, we proceed with the formal analysis.

II. The Basic Equations

Parker (1968) has given the two-fluid equations which govern the behavior of a system of thermal gas, cosmic-gas, and magnetic field in the approximations that the small-scale irregularities of the magnetic field can be ignored, and that the motions are well-coupled in the two directions perpendicular to the magnetic field, while they are completely decoupled in the direction parallel to the field [2]). The generalization of the dynamical equations, for slow bulk motions of the cosmic-ray gas, to include the relativistic contribution to its inertia can be formulated along the lines given by Lerche and Parker (1966). The gravitational field of the Galaxy is also easily incorporated, but the direct action of gravity is entirely negligible for the dynamics of the cosmic-ray gas. The cosmic-rays are confined to the Galaxy by the interstellar magnetic field, in turn, is held down by the weight of the thermal gas (Biermann and Lust, 1960; Parker, 1966).

To close the set of equations numbered (11)–(15) by Parker (1968), when generalized to include the effects described above, we need to write equations which govern the behavior of the thermal and cosmic-ray pressures, P and P_{cr}. For the thermal gas, we follow Parker and heuristically adopt the following variation of P with variations of ϱ:

$$\frac{DP}{Dt} = \gamma \frac{P}{\varrho} \frac{D\varrho}{Dt}, \tag{1}$$

where $D/Dt = \partial/\partial t + \boldsymbol{v} \cdot \nabla$ is the substantive derivative associated with the motion of the thermal gas, and where γ is a constant. Although Eq. (1) has the same form as the energy equation of an ideal gas (i.e., a gas limited to adiabatic variations), we allow γ to be dif-

[2]) The last assumption has come into question following the work of Lerche (1966, 1967d), Wentzel (1968, 1969), Kulsrud and Pierce (1969), and Tademaru (1969). These authors, among others, have shown that the streaming velocity of the bulk of the high-energy cosmic-rays with respect to the thermal gas cannot much exceed the Alfvén speed of the *ionized* component of the interstellar gas. Since this speed is of order $10^2\,\mathrm{km\,s^{-1}}$ and since the large-scale hydromagnetic wave speeds are of order $10\,\mathrm{km\,s^{-1}}$, we can ignore the coupling of the two "fluids" in the direction parallel to the magnetic field until highly nonlinear processes begin to demand the development of a slip of about $10^2\,\mathrm{km\,s^{-1}}$.

ferent from the ratio of specific heats to simulate the effects of radiative cooling (or inelastic cloud collisions if we visualize P as the large-scale "pressure" associated with random cloud motions). In these circumstances $\gamma \lesssim 1$ represents an appropriate choice (see also Ames, 1973).

If we were interested in examining high-frequency waves in the cosmic-ray gas, we would need to write down an equation similar to Eq. (1) for P_{cr}. For the timescales of interest to us, however, the small-amplitude response of the cosmic-ray gas in the direction parallel to the magnetic field is nearly instantaneous since it has an associated sound speed which is close to the speed of light (Parker, 1965). For low frequencies and slow bulk motions, we may ignore the effective inertial density of the cosmic-ray gas so that the independent conservation of the momentum of the cosmic-ray gas in the direction parallel to the magnetic field can be expressed as

$$\boldsymbol{B} \cdot \nabla P_{cr} = 0 .$$

This simple equation, derived solely from dynamical considerations, provides the requisite constraint on P_{cr}. It corresponds to the approximation that the cosmic-ray pressure remains constant along each tube of magnetic force, i.e., that the suprathermal mode discussed by Parker (1965) propagates with infinite speed rather than "just" with a speed generally close to the speed of light (see Footnote 5).

A complete set of dynamical equations (we assume the gravitational field of the Galaxy, $-\nabla\mathscr{V}$, is specified) is now given, in an obvious notation, by the vector invariant set

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{v}) = 0 , \tag{2a}$$

$$\varrho\left[\frac{\partial \boldsymbol{v}}{\partial t} + \nabla(\tfrac{1}{2}|\boldsymbol{v}|^2) + (\nabla\times\boldsymbol{v})\times\boldsymbol{v} + \nabla\mathscr{V}\right] + \nabla(P+P_{cr}) - \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = 0 , \tag{2b}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla\times(\boldsymbol{B}\times\boldsymbol{v}) = 0 , \tag{2c}$$

$$\frac{\partial P}{\partial t} + \boldsymbol{v}\cdot\nabla P + \gamma P \nabla\cdot\boldsymbol{v} = 0 , \tag{2d}$$

$$\boldsymbol{B}\cdot\nabla P_{cr} = 0 . \tag{2e}$$

where we have used the equation of continuity of the thermal gas (2a) to eliminate ϱ from Eq. (1). As an aside, note that if $\boldsymbol{B}=0$, the cosmic-ray gas is decoupled from the system in all three directions, and ∇P_{cr} should not be included in Eq. (2b).

Previous workers have used the equation

$$\frac{\partial P_{cr}}{\partial t} + \boldsymbol{v}\cdot\nabla P_{cr} = 0 \tag{3}$$

instead of Eq. (2e) to determine P_{cr}. This is generally an invalid method which obscures the reason for the formal presence of the gradient of the cosmic-ray pressure in Eq. (2b) for the direction parallel to the magnetic field. However, in agreement with Parker's (1966) assertion and with Ames' (1973) analysis, we find that the two Eqs. (2e) and (3) are identical in the linearized regime for special geometries. Obviously, any nonlinear calculation should adopt Eq. (2e) – at least until the effects described by Kulsrud and Pierce (1969) and by Wentzel (1969) become important. This point is especially important for the interpretation of the enhancement of synchrotron emission observed in the spiral arms of external galaxies (see the following paper).

a) The Equilibrium State

We now adopt cylindrical coordinates (ϖ, θ, z) with $\varpi=0$ describing the axis of rotation and $z=0$ describing the principal plane of the disk. We ignore the self-gravity of the gas, and we assume a fixed gravitational field for the Galaxy idealized to have the cylindrical components

$$-\nabla\mathscr{V} = (-f(\varpi), 0, -g\,\mathrm{sign}(z)) \tag{4}$$

where g is assumed to be a constant[3]). A possible equilibrium state consistent with the above gravitational field is given by a field of differential rotation and an azimuthal magnetic field (subscript "0" denotes the equilibrium state)

$$\boldsymbol{v}_0 = (0, \varpi\Omega(\varpi), 0), \quad \boldsymbol{B}_0 = (0, B_0(z), 0) , \tag{5a}$$

$$P_0 = a^2\varrho_0(z) , \quad \frac{B_0^2}{8\pi} = \alpha a^2 \varrho_0(z) , \quad P_{cr_0} = \beta a^2 \varrho_0(z) , \tag{5b}$$

where a, α, and β are assumed to be constants. We refer to a as the "thermal velocity" but it includes the contribution of the random cloud motions to the large-scale pressure. The speed at which acoustic waves propagate is $\gamma^{1/2}a$; Alfvén waves, $(2\alpha)^{1/2}a$. Generally, α and β will be taken to be of order unity.

Note that

$$\nabla\cdot\boldsymbol{B}_0 = 0 \quad\text{and}\quad \nabla\times\boldsymbol{B}_0 = -\boldsymbol{e}_\varpi \frac{dB_0}{dz} + \boldsymbol{e}_z \frac{B_0}{\varpi} , \tag{6}$$

so that $\boldsymbol{B}_0$ requires primarily a radial current to maintain it if the vertical scale height is small in comparison with the radius of the disk. Since we are not concerned here with the origin of the interstellar magnetic field, we shall leave the origin of the radial current unspecified. In any case, a large-scale average field directed primarily in the azimuthal direction seems to be consistent with the existing observations (Manchester, 1972). Moreover,

[3]) A linear variation of g with z would be more realistic, but the price for this realism is greatly increased complexity for the mathematical analysis (see Parker, 1966). A slow variation of f with z, and slow variations of g, a, α, β, γ with ϖ are more easily incorporated, but the equations then become littered with terms which would eventually be dropped in the "local" analysis to be adopted later. To keep the perturbation analysis as simple as possible, we therefore adopt the simplest equilibrium state consistent with a field of differential rotation. For precise comparisons with observations, however, it is advisable to think in terms of gentle variations of g, a, α, β with position in the Galaxy, and it may be especially advisable to remember the z-variation of g (see Appendix).

given the strong differential rotation which exists in the Galaxy, it is difficult to see theoretically – even in the context of the density-wave theory for spiral structure (Lin *et al.*, 1969) – how such a situation could be realistically violated to any large extent on a global scale. A possible reversal of the magnetic field across the midplane, i.e., a change in the sign of $B_0(z)$ for $z \gtrless 0$, will not affect any of our results since the normal modes can be classed into two distinct parities (§ IIb), and the solutions can be described in terms of the behavior above the midplane $z=0$.

The dynamical equations now place the following two constraints on the equilibrium

$$\varpi \Omega^2(\varpi) = f(\varpi) + \frac{1}{\varpi}\frac{B_0^2(z)}{4\pi \varrho_0(z)} = f(\varpi) + \frac{2\alpha a^2}{\varpi}, \quad (7a)$$

$$(1+\alpha+\beta)\frac{a^2}{\varrho_0}\frac{d\varrho_0}{dz} = -g\,\mathrm{sign}(z). \quad (7b)$$

Equation (7b) can be integrated to give

$$\frac{\varrho_0(z)}{\varrho_0(0)} = \frac{B_0^2(z)}{B_0^2(0)} = \frac{P_0(z)}{P_0(0)} = \frac{P_{cr_0}(z)}{P_{cr_0}(0)} = \exp\left(-\frac{|z|}{H}\right), \quad (8)$$

where H, the equivalent isothermal scale height, is given by

$$H = (1+\alpha+\beta)\,a^2/g. \quad (9)$$

b) Linearized Perturbation Equations

We wish to examine the stability of the assumed equilibrium state to small perturbations, and we are interested in the full three-dimensional problem. To remove dissimilar z-variations, we express the perturbations as fractions of some appropriate equilibrium quantity:

$$\varrho = \varrho_0(1+s), \quad v_\varpi = a u_\varpi, \quad v_\theta = \varpi\Omega + a u_\theta, \quad v_z = a u_z, \quad (10a)$$

$$B_\varpi = B_0 b_\varpi, \quad B_\theta = B_0(1+b_\theta), \quad B_z = B_0 b_z, \quad (10b)$$

$$P = P_0(1+p), \quad P_{cr} = P_{cr_0}(1+p_{cr}). \quad (10c)$$

The dimensionless perturbation variables s, $\boldsymbol{u}$, $\boldsymbol{b}$, p, and p_{cr} are generally function of (ϖ, θ, z, t) and will be assumed to be infinitesimal in magnitude for the analysis below. The linearized equations governing the perturbations are easily expressed in cyclindrical coordinates; however, we shall not record the results here since we shall introduce their more manageable approximate forms in § III. We shall merely remark that the use of the constraint $\nabla \cdot \boldsymbol{B} = 0$ allows the linearized version of the θ-component of the field-freezing equation to be expressed as

$$\frac{\partial b_\theta}{\partial t} + \Omega\frac{\partial b_\theta}{\partial \theta} - \varpi\frac{d\Omega}{d\varpi}b_\varpi + a\frac{\partial u_\varpi}{\partial \varpi} - \varepsilon\frac{a}{2H}u_z + a\frac{\partial u_z}{\partial z} = 0 \quad (11)$$

where $\varepsilon = \mathrm{sign}(z)$; whereas if we operate on the linearized version of Eq. (2e) by $(\partial/\partial t + \Omega\partial/\partial\theta)$ and use the linearized version of the z-component of the field-freezing equation to eliminate b_z, and if we integrate the resultant equation in θ, we obtain

$$\frac{\partial p_{cr}}{\partial t} + \Omega\frac{\partial p_{cr}}{\partial \theta} - \varepsilon\frac{a}{H}u_z = 0. \quad (12)$$

We easily verify that the above equation could also have been obtained by linearizing Eq. (3), in accordance with our previous remarks. Although Eq. (12) involves one unnecessary differentiation in time in comparison with the linearized version of Eq. (2e), we choose to use the former because it possesses a mathematical structure more similar to the form of the other dynamical equations; this will result in a considerable saving of notation in the later development.

III. The Local Approximation

Apart from the sign of z, the coefficients in the linearized dynamical equations for the variables s, $\boldsymbol{u}$, $\boldsymbol{b}$, p, p_{cr} depend only on ϖ; moreover, these coefficients vary appreciably only over distances which are comparable with the total radial extent of the disk. Since we are interested in disturbances with reciprocal wavenumbers characteristically of order H, which is smaller than ϖ by a factor of about 50, it is tempting simply to evaluate all of the coefficients at one fixed radius, ϖ_0, and to perform a normal-mode analysis by the usual method of separation of variables. This represents a valid procedure for all terms except those typified by the combination $\partial q/\partial t + \Omega\,\partial q/\partial\theta$ where q represents the column vector $\{s, u_\varpi, u_\theta, u_z, b_\varpi, b_\theta, b_z, p, p_{cr}\}$. If we assume a (θ, t) dependence for q of the form $\exp[i(\omega t - m\theta)]$, with m equal to an integer, we always obtain the eigenfrequency ω in the combination

$$i(\omega - m\Omega) \simeq -\omega_I + i[\omega_R - m\Omega(\varpi_0)] - im(\varpi - \varpi_0)\frac{d\Omega}{d\varpi}(\varpi_0), \quad (13)$$

where we have assumed that $\omega = \omega_R + i\omega_I$ can generally be complex and where we have carried out a Taylor series expansion of $\Omega(\varpi)$ about $\varpi = \varpi_0$.

The problem develops because we are generally interested in growing disturbances, characterized by an azimuthal wavenumber corresponding to $m = 0(\varpi/H) \gg 1$, which are basically carried along with the equilibrium flow, i.e., in nearly corotating disturbances where $\omega_R - m\Omega(\varpi_0)$ and the growth rate, $-\omega_I$, are comparable to a/H. Thus, the variation of $\Omega(\varpi)$ in $(\omega - m\Omega)$ is negligible only if we restrict our attention to that portion of space where

$$|\varpi - \varpi_0| \ll a|(\varpi\, d\Omega/d\varpi)_0|^{-1}. \quad (14)$$

Now, strong differential rotation implies $\varpi d\Omega/d\varpi = 0(\Omega)$, whereas the parameter $a/H\Omega$ should be treated as being of order unity (it is about 1.5 in the solar neighborhood). Thus, the right-hand side of the inequality (14) is of order H, and the neglect of the variation of Ω in $(\omega - m\Omega)$ would restrict the validity of the analysis to a radial region much less than one vertical scale height! (Actually, to radial wavenumbers $\gg H^{-1}$.) Although it is difficult to consider the full effects of shear for large azimuthal wavenumbers in an analytical study, we shall formulate the problem to obtain an extended regime of validity which should prove useful in future numerical work. We adopt a "local approximation" which deals with $\Omega(\varpi)\,\partial q/\partial\theta$ by expanding $\Omega(\varpi)$ about $\varpi = \varpi_0$ and truncating the Taylor series after the linear term. This simple device, which leads to an elegant method for treating localized waves in a shearing flow-field, was invented independently by Goldreich and Lynden-Bell (1965b) and by Julian and Toomre (1966) in other contexts[4]). Motivated by these remarks, we now introduce the transformation to dimensionless Cartesian coordinates $(\varpi, \theta, z, t) \to (x', y', z', \tau)$

$$x' = \frac{1}{H}(\varpi - \varpi_0), \quad y' = \frac{\varpi_0}{H}[\theta - \Omega(\varpi_0)\,t], \quad z' = \frac{z}{H}, \quad \tau = \frac{at}{H}. \tag{15}$$

The various partial derivatives transform as

$$\frac{\partial}{\partial\varpi} = \frac{1}{H}\frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial\theta} = \frac{\varpi_0}{H}\frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{1}{H}\frac{\partial}{\partial z'},$$
$$\frac{\partial}{\partial t} = \frac{a}{H}\frac{\partial}{\partial\tau} - \frac{\varpi_0}{H}\Omega(\varpi_0)\frac{\partial}{\partial y'}, \tag{16}$$

and we are especially careful to carry out an expansion for

$$\Omega(\varpi)\frac{\partial}{\partial\theta} = \frac{\varpi_0}{H}\Omega(\varpi_0)\frac{\partial}{\partial y'} + x'\varpi_0\frac{d\Omega}{d\varpi}(\varpi_0)\frac{\partial}{\partial y'}. \tag{17}$$

In what follows, we shall drop the cumbersome prime notation except when there might be some confusion about z and z'. The perturbation velocity and magnetic field transform as $(u_\varpi, u_\theta, u_z) \to (u_x, u_y, u_z)$, $(b_\varpi, b_\theta, b_z) \to (b_x, b_y, b_z)$, and, for reference, we note that the circular velocity $\varpi\Omega$ and equilibrium magnetic field $\boldsymbol{B}_0$ are now directed in the y-direction whereas the vertical gravity g is still along z.

To complete our nondimensionalization, we define the parameters

$$\Lambda_1 = 2\Omega(\varpi_0)\,H/a, \quad \Lambda_2 = \kappa^2(\varpi_0)\,H/2\Omega(\varpi_0)\,a, \tag{18}$$

where $\varkappa^2$ is the square of the epicyclic frequency

$$\varkappa^2 = \frac{2\Omega}{\varpi}\frac{d}{d\varpi}(\varpi^2\Omega) \tag{19}$$

and is positive if the angular momentum distribution of the gas (per unit mass) $\varpi^2\Omega$ increases outwards; the latter is Rayleigh's criterion for the stability of differentially-rotating fluids (see, e.g., Chandrasekhar, 1961). We note in passing that $(\Lambda_1 - \Lambda_2) = -a^{-1}H \cdot (\varpi\, d\Omega/d\varpi)_0$, with $(\varpi\, d\Omega/d\varpi)_0$ equal to twice Oort's constant A, is a dimensionless measure of the local shear. With these definitions and with the adoption of the local approximation, $\varpi_0/H \to \infty$, we can simplify the linearized dynamical equations, expressed in an accurate form in cyclindrical coordinates, to obtain the following dimensionless set which is valid for all z. [In what follows we have used $a^{-2}H(f - \varpi\Omega^2) = -2\alpha H/\varpi \to 0$ and $a^{-2}Hg = (1 + \alpha + \beta)$ to eliminate f and g.]

$$\frac{\partial s}{\partial\tau} - x(\Lambda_1 - \Lambda_2)\frac{\partial s}{\partial y} - \varepsilon u_z + \frac{\partial u_z}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0, \tag{20a}$$

$$\frac{\partial u_x}{\partial\tau} - x(\Lambda_1 - \Lambda_2)\frac{\partial u_x}{\partial y} - \Lambda_1 u_y + \frac{\partial p}{\partial x} + \beta\frac{\partial p_{cr}}{\partial x} - 2\alpha\left(\frac{\partial b_x}{\partial y} - \frac{\partial b_y}{\partial x}\right) = 0, \tag{20b}$$

$$\frac{\partial u_y}{\partial\tau} - x(\Lambda_1 - \Lambda_2)\frac{\partial u_y}{\partial y} + \Lambda_2 u_x + \frac{\partial p}{\partial y} + \beta\frac{\partial p_{cr}}{\partial y} + \varepsilon\alpha b_z = 0, \tag{20c}$$

$$\frac{\partial u_z}{\partial\tau} - x(\Lambda_1 - \Lambda_2)\frac{\partial u_z}{\partial y} + \varepsilon(1 + \alpha + \beta)\,s - \varepsilon(p + \beta p_{cr}) + \frac{\partial p}{\partial z} + \beta\frac{\partial p_{cr}}{\partial z} - 2\alpha\left(\frac{\partial b_z}{\partial y} + \varepsilon b_y - \frac{\partial b_y}{\partial z}\right) = 0, \tag{20d}$$

$$\frac{\partial b_x}{\partial\tau} - x(\Lambda_1 - \Lambda_2)\frac{\partial b_x}{\partial y} - \frac{\partial u_x}{\partial y} = 0, \tag{20e}$$

$$\frac{\partial b_y}{\partial\tau} - x(\Lambda_1 - \Lambda_2)\frac{\partial b_y}{\partial y} + (\Lambda_1 - \Lambda_2)\,b_x + \frac{\partial u_x}{\partial x} - \frac{\varepsilon}{2}u_z + \frac{\partial u_z}{\partial z} = 0, \tag{20f}$$

$$\frac{\partial b_z}{\partial\tau} - x(\Lambda_1 - \Lambda_2)\frac{\partial b_z}{\partial y} - \frac{\partial u_z}{\partial y} = 0, \tag{20g}$$

$$\frac{\partial p}{\partial\tau} - x(\Lambda_1 - \Lambda_2)\frac{\partial p}{\partial y} - \varepsilon u_z + \gamma\left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right] = 0, \tag{20h}$$

$$\frac{\partial p_{cr}}{\partial\tau} - x(\Lambda_1 - \Lambda_2)\frac{\partial p_{cr}}{\partial y} - \varepsilon u_z = 0. \tag{20i}$$

[4]) For small m, the "local approximation" is more restrictive than the WKBJ approximation introduced by Lin and Shu (1964) which allows arbitrary (but smooth) variations of $\Omega(\varpi)$, $\varkappa(\varpi)$, etc., and which allows successive approximations to be made to improve the accuracy of the asymptotics, e.g., to derive information concerning the variation of the wave amplitude (Shu, 1970).

Note that the "Coriolis force" in Eq. (20c) enters with $\Lambda_2 \propto \varkappa^2/2\Omega$ replacing $\Lambda_1 \propto 2\Omega$; this result for differentially-rotating disks is well known from work on the density-wave theory of spiral structure.

We wish to perform a linearized stability analysis for the system using the above model equations. There exists two standard methods for performing such an analysis – the initial-value problem and the normal-modes approach. The latter gives better insight into the physical processes, but the former is more appropriate for an unbounded medium, which our system has become in the (x, y) directions by the adoption of the approximation $\varpi_0/H \to \infty$. The normal modes in an unbounded or semi-bounded medium usually form a continuum. For a continuum, an individual mode loses much of its usual significance because a general disturbance will not usually excite a single mode. We shall formulate the problem as an initial-value problem and then recover from this formulation the condition for normal modes; the formalism turns out to be non-trivial because of the spatial dependence on x.

a) The Initial-value Problem

To minimize repetitious definitions, we let q represent the column vector $\{s, u_x, u_y, u_z, b_x, b_y, b_z, p, p_{cr}\}$, and we introduce the Fourier-transform pair Q and $e^{-k|z|}q$ defined through

$$Q(\xi,\eta,\zeta,\tau;k) = \left(\frac{1}{2\pi}\right)^3 \iiint_{-\infty}^{\infty} e^{-k|z|}\, q(x,y,z,\tau) \cdot e^{i(\xi x+\eta y+\zeta z)}\, dx\, dy\, dz\,, \tag{21a}$$

$$q(x,y,z,\tau) = e^{k|z|} \iiint_{-\infty}^{\infty} Q(\xi,\eta,\zeta,\tau;k) \cdot e^{-i(\xi x+\eta y+\zeta z)}\, d\xi\, d\eta\, d\zeta\,. \tag{21b}$$

where the wavenumbers ξ, η, ζ, and the extra parameter k are all real, and where Q represents the column vector $\{S, U_x, U_y, U_z, \mathscr{B}_x, \mathscr{B}_y, \mathscr{B}_z, \Pi, \Pi_{cr}\}$. The multiplicative factor $\exp(-k|z|)$ is included in Eq. (21a) to suppress any exponential growth in amplitude of the perturbation quantity q for large $|z|$ – which constitutes, as we shall see, a natural aspect of wave propagation in an exponential atmosphere (cf. Lerche and Parker, 1967).

For the present, we allow k to be a free parameter whose value is to be chosen sufficiently positive to guarantee the convergence of the integral transform (21a); thus, the transform of q in the z-coordinate is essentially a Laplace transform.

For notational simplicity, we shall suppress explicit display of the parametric dependence on k, and we shall henceforth write $Q(\xi,\eta,\zeta,\tau)$ to mean $Q(\xi,\eta,\zeta,\tau;k)$. To be concise, we shall also use the definition

$$\mu = \varepsilon k - i\zeta\,, \tag{22}$$

with $\varepsilon = \mathrm{sign}(z)$, and we shall refer to $i\mu$ as the "complex wavenumber" associated with the z-direction.

Upon substituting Eq. (21b) into Eqs. (20a)–(20i) and noting that the transform of $x\,\partial q/\partial y$ is $-\eta\,\partial Q/\partial \xi$, we obtain the following set of linear partial-differential equations in the independent variables (ξ,η,ζ,τ):

$$\frac{\partial Q_j}{\partial \tau} + \eta(\Lambda_1 - \Lambda_2)\frac{\partial Q_j}{\partial \xi} + A_{jl}Q_l = 0\,, \quad j = 1, 2, \ldots, 9\,, \tag{23}$$

where Q_j is the j-th component of the column vector Q and where we have adopted Einstein's summation convention with respect to repeated indices – the summation in l extending from 1 to 9. The matrix A is given in Fig. 1. Note that A has no nonzero diagonal terms and that it depends on the real wavenumbers ξ, η, ζ, and on the parameters α, β, γ, ε, k, Λ_1, Λ_2. Henceforth, we shall suppress the parametric dependences and write $A(\xi,\eta,\zeta)$ to denote the matrix illustrated in Fig. 1.

If we regard the set of transform Eqs. (23) to be a coupled set of linear partial-differential equations in the variables (ξ,η,ζ,τ), we may write down the associated characteristic equations as (see, e.g., Garabedian, 1964)

$$\frac{d\xi}{d\tau} = \eta(\Lambda_1 - \Lambda_2), \quad \frac{d\eta}{d\tau} = 0, \quad \frac{d\zeta}{d\tau} = 0, \qquad \frac{dQ_j}{d\tau} = -A_{jl}Q_l\,. \tag{24}$$

The first three equations can be integrated to yield

$$\xi(\tau) = \eta(\Lambda_1 - \Lambda_2)\,\tau + \xi_0, \qquad \eta = \text{constant}, \quad \zeta = \text{constant}\,, \tag{25}$$

$$\begin{vmatrix}
0 & -i\xi & -i\eta & \mu-\varepsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\Lambda_1 & 0 & 2i\alpha\eta & -2i\alpha\xi & 0 & -i\xi & -i\beta\xi \\
0 & \Lambda_2 & 0 & 0 & 0 & 0 & \varepsilon\alpha & -i\eta & -i\beta\eta \\
\varepsilon(1+\alpha+\beta) & 0 & 0 & 0 & 0 & 2\alpha(\mu-\varepsilon) & 2i\alpha\eta & \mu-\varepsilon & \beta(\mu-\varepsilon) \\
0 & i\eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i\xi & 0 & \mu-\varepsilon/2 & \Lambda_1-\Lambda_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i\eta & 0 & 0 & 0 & 0 & 0 \\
0 & -i\gamma\xi & -i\gamma\eta & \gamma\mu-\varepsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon & 0 & 0 & 0 & 0 & 0
\end{vmatrix}$$

Fig. 1. The matrix A; see text for definition of symbols

where ξ_0 is a constant along the characteristics. The general solution for Q_j is now obtained by integration

$$Q_j(\xi(\tau), \eta, \zeta, \tau) = \mathcal{Q}_j(\xi_0, \eta, \zeta) - \int_0^\tau A_{jl}(\xi(\tau'), \eta, \zeta) \cdot Q_l(\xi(\tau'), \eta, \zeta, \tau')\, d\tau', \tag{26}$$

where the "constant" of integration $\mathcal{Q}_j$ is an arbitrary function of its arguments. We identify $\xi(\tau)$ with ξ so that the Eulerian form of the general solution reads

$$Q_j(\xi, \eta, \zeta, \tau) = \mathcal{Q}_j(\xi - \eta(\Lambda_1 - \Lambda_2)\tau, \eta, \zeta) - \int_0^\tau \Lambda_{jl}(\xi - \eta(\Lambda_1 - \Lambda_2)(\tau - \tau'), \eta, \zeta) \cdot Q_l(\xi - \eta(\Lambda_1 - \Lambda_2)(\tau - \tau'), \eta, \zeta, \tau')\, d\tau'. \tag{27}$$

The arbitrary function $\mathcal{Q}_j$ now has an immediate physical interpretation,

$$\mathcal{Q}_j(\xi, \eta, \zeta) = Q_j(\xi, \eta, \zeta, 0), \tag{28}$$

the Fourier transform of the initital column vector $e^{-k|z|} q_j(x, y, z, 0)$.

Equations (27) and (28) constitute the complete formal solution of the general initial-value problem. The solution is only a formal one because the integral on the right-hand side involves the unknown variables Q_j evaluated at times *previous* to τ and at displaced wavenumbers from ξ. The coupled set of integral equations is, however, particularly suited for numerical solution on an electronic computer. The schematic steps are the following. Given the initial values $e^{-k|z|} q_j(x, y, z, 0)$ throughout space (consistent with $\nabla \cdot \boldsymbol{B} = 0$ and $\boldsymbol{B} \cdot \nabla P_{cr} = 0$), satisfying appropriate boundary conditions at infinity (i.e., square-integrability), obtain their Fourier transforms $Q_j(\xi, \eta, \zeta, 0)$. At each grid point (ξ, η, ζ), set $\mathcal{Q}_j(\xi, \eta, \zeta) = Q_j(\xi, \eta, \zeta, 0)$. The solution at any later time τ can be obtained by step-by-step numerical integration of Eq. (27). The transformation back to spatial coordinates can be undertaken by fast transform techniques now available for machine computation.

b) *Shearing Wavelets and Normal Modes*

Because the combination $\xi - \eta(\Lambda_1 - \Lambda_2)\tau$ appears as a characteristic coordinate in the above analysis as well as in the earlier work of Julian and Toomre (1966, see also Goldreich and Lynden-Bell, 1965b), there exists a great temptation to endow special physical significance on disturbances in which

$$\xi - \eta(\Lambda_1 - \Lambda_2)\tau = \text{constant}, \tag{29}$$

i.e., on waves whose fronts shear with the local material – "shearing wavelets" in the nomenclature of Julian and Toomre.

Our interpretation, however, agrees more closely with Hunter's (1972, 1973) point of view. The coupling of the different Fourier components in ξ is a reflection of the model being spatially inhomogeneous in the x-direction. Therefore, purely sinusoidal functions do not represent the most natural set of functions in which to expand for that direction. If one chooses, of course, one can expand in any complete set of functions and superimpose (as Julian and Toomre did), but the usual price is a coupling of the transformed equations in wavenumber space – or in frequency space if the spatial variations are transformed into time variations as was done in the study of Goldreich and Lynden-Bell.

True normal mode solutions, in the usual sense of the phrase, have an exponential or oscillatory behavior in time, i.e.,

$$Q_j(\xi, \eta, \zeta, \tau) = Q_j^{(n)}(\xi)\, e^{n\tau}, \tag{30}$$

where n is the dimensionless growth rate, which may be complex, and where for notational simplicity we have suppressed explicit display of the (parametric) dependences on η and ζ since Fourier components with different (η, ζ) are not interrelated. We may obtain the condition on the normal mode solutions, if any exist, from the initial-value formulation by considering the $\tau \to +\infty$ behavior under the formal assumption $\mathrm{Re}(n) > 0$ (see, e.g., Landau, 1946). The substitution of Eq. (30) into Eq. (27) yields

$$Q_j^{(n)}(\xi) = e^{-n\tau}\mathcal{Q}_j - \int_0^\tau [\Lambda_{jl} Q_l^{(n)}](\xi - \eta(\Lambda_1 - \Lambda_2)(\tau - \tau')) \cdot e^{-n(\tau - \tau')}\, d\tau', \tag{31}$$

where the braces followed by the parenthesis indicates that the column vector $A_{jl} Q_l^{(n)}$ is to be evaluated at the ξ-wavenumber enclosed in the parenthesis. If we use $\tau'' = \tau - \tau'$ instead of τ' as the integration variable and take the limit $\tau \to +\infty$, we obtain the requirement

$$Q_j^{(n)}(\xi) + \int_0^\infty [\Lambda_{jl} Q_l^{(n)}](\xi - \eta(\Lambda_1 - \Lambda_2)\tau'') \cdot e^{-n\tau''}\, d\tau'' = 0, \tag{32}$$

which constitutes a set of coupled linear homogeneous integral equations to solve for the eigenfunctions $Q_j^{(n)}$ and the eigenvalue n. The condition on the normal modes for $\mathrm{Re}(n) \leqq 0$ can be obtained from Eq. (32) by analytic continuation.

Note that the form of the integral in Eq. (32) is that of a Laplace transform; however, the Laplace transform is not taken on the original Eqs. (23) but on them in their characteristic form. There is one important limiting case where the coupled set (32) can be easily solved by analytical methods, but before we discuss this limit (the "weak shear limit"), we make some remarks concerning boundary conditions.

The structure of Eqs. (20) allows solutions to be uncoupled in terms of their parity in z. Thus, without loss of generality, we may look for normal-mode solutions (possibly with discontinuous z-derivatives) in which (A)

s, u_x, u_y, b_x, b_y, p, and p_{cr} are even in z while u_z and b_z are odd, and (B) *vice-versa*. Thus, for example, for the problem of normal modes of class A, we may impose the vertical boundary conditions,

$$u_z = b_z = 0 \quad \text{at} \quad z = 0\,; \tag{33a}$$

$$s, \boldsymbol{u}, \boldsymbol{b}, p, p_{cr} \text{ exponentially bounded by } e^{|z|/2} \text{ as } |z| \to \infty\,; \tag{33b}$$

and concentrate on the semi-bounded region $z > 0$ where $\varepsilon = +1$. Condition (33a) really imposes only a single constraint on the solution since equation (20g) guarantees $b_z = 0$ at $z = 0$ if $u_z = 0$. Condition (33b) requires that various quadratic energy densities such as $\varrho_0 |\boldsymbol{u}|^2/2$, $B_0^2 |\boldsymbol{b}|^2/8\pi$, $\gamma P_0 s^2/2$, $P_{cr_0} p_{cr} s$ remain bounded at infinity. Analagous considerations apply to normal modes of class B. The class of perturbations satisfying Eqs. (33) are clearly included in our formulation of the initial-value problem (§ IIIa); therefore, if we actually carried out an initial-value calculation using our formalism, the *eventual* time development will automatically lead to a selection, within the context of the linear theory, of the normal mode with the largest growth rate (cf. Lerche and Parker, 1967).

In terms of the constraints placed on the complex wavenumber, $i\mu$, Eq. (33b) requires $k = \mathrm{Re}(\mu) \leqq 1/2$, whereas Eq. (33a) requires that solutions with $\zeta = -\mathrm{Im}(\mu)$ positive can be paired with solutions with ζ negative in such a way that

$$e^{n\tau} e^{-i\eta y} e^{k|z|} \sin(|\zeta|\, z) \int_{-\infty}^{\infty} U_z^{(n)}(\xi)\, e^{-i\xi x}\, d\xi \tag{34}$$

represents a possible solution for u_z (standing wave in the z-direction). This constraint is nontrivial because it demands that the eigenvalue n does not distinguish between the two possible choices of sign for ζ (see also Parker, 1967b). The constraint that n does not depend on the sign of ζ applies also to class B perturbations. This point is important to realize right at the outset whenever one allows complex wavenumbers and one is interested in looking for unstable waves (complex frequencies). In such cases, one must apply appropriate boundary conditions to avoid solutions which extract their growth in time at the expense of an artificial attenuation in space. (Such a situation corresponds to the placement of a nonsteady emitter of waves behind one of the boundaries – in the present case, the mid-plane $z = 0$.)

IV. Normal Modes in the Weak-shear Limit

The set of integral Eqs. (32) for the normal modes can be solved easily in one limit, namely, the limit $\eta(\Lambda_1 - \Lambda_2)/\xi n \to 0$. We refer to this limit as the "weak-shear limit" because it corresponds physically to the consideration of waves whose growth rates (or frequencies) are large in comparison with the kinematic shear rate. (For reference, we note $(\Lambda_1 - \Lambda_2) \simeq 0.8$ in the solar neighborhood.) The limitation of our attention to waves which satisfy this condition is obviously overly restrictive for the general problem; however, we shall see that it suffices for the demonstration of one of the central results of this paper – namely, that no finite amount of rotation or shear can completely stabilize Parker's mode if the system has the assumed geometry.

For $|\eta(\Lambda_1 - \Lambda_2)/\xi n| \ll 1$, we may formally carry out a series expansion of Eq. (32) to obtain

$$n Q_j + A_{jl} Q_l + \sum_{N=1}^{\infty} (-)^N [\eta(\Lambda_1 - \Lambda_2)/\xi n]^N \xi^N \cdot \frac{\partial^N}{\partial \xi^N} [A_{jl} Q_l] = 0\,, \tag{35}$$

where we have dropped the cumbersome superscript n notation denoting eigenvectors, and where all of the above quantities are to be evaluated at (ξ, η, ζ). In the limit $\eta(\Lambda_1 - \Lambda_2)/\xi n \to 0$, we obtain the matrix equation,

$$(nI + A)\, Q = 0\,, \tag{36}$$

where I is the identity matrix. Equation (36) could, of course, have been obtained much more directly by noting that $\eta(\Lambda_1 - \Lambda_2) = 0$ decouples the ξ-wavenumbers in Eq. (23). We prefer the present derivation because it demonstrates that decoupling requires only that $|\eta(\Lambda_1 - \Lambda_2)/\xi n| \ll 1$ and not $\Lambda_1 = \Lambda_2$ (uniform rotation) or $\eta = 0$ ("axisymmetric" disturbances). Retaining the possibility that $\Lambda_1 \neq \Lambda_2 \neq 0$, and $\eta \neq 0$, we refer to Eq. (36) as the "weak-shear" limit of the normal-mode equations.

For Eq. (36) to possess nontrivial solutions, the determinant of the coefficient matrix must be zero. This condition leads to a dispersion relation, the analysis of which essentially occupies the rest of this paper.

a) Dispersion Relation in the Weak-shear Limit

The condition $\det(nI + A) = 0$, where I is the identity matrix and A is the matrix shown in Fig. 1 with $\varepsilon = +1$, leads, after considerable tedious algebra, to the relation

$$n^2 \Delta(n) = 0\,, \tag{37}$$

where $\Delta(n)$ is the polynomial

$$\Delta(n) = n^7 + Bn^5 + Cn^4 + Dn^3 + En^2 + Fn + G\,, \tag{38}$$

and where the coefficients $B, \dots, G$ are given by

$$B = \Lambda_1 \Lambda_2 + b + 2\alpha\eta^2\,, \tag{39a}$$

$$C = (\gamma + 2\alpha)(\Lambda_1 - \Lambda_2)\,\xi\eta\,, \tag{39b}$$

$$D = \Lambda_1 \Lambda_2 [2\alpha\eta^2 + (\gamma + 2\alpha)\,\mu(1-\mu)] + 2\alpha\eta^2 b + c + 2\alpha(1 + \alpha + \beta)\,\xi^2\,, \tag{39c}$$

$$E = \xi\eta\,\{(\Lambda_1 - \Lambda_2)\,[(\gamma + 2\alpha)\,2\alpha\eta^2 + 2\alpha\gamma(\eta^2 - \eta_P^2) + 2\alpha(1 + \alpha + \beta)] + 2\alpha(1 + \alpha + \beta)\,\Lambda_1(1 - 2\mu)\}\,, \tag{39d}$$

$$F = 2\alpha\eta^2 \{\Lambda_1(\Lambda_1-\Lambda_2)[(1+\alpha+\beta)(1-\mu) - \gamma\mu(1-\mu)] + c\}, \tag{39e}$$

$$G = 2\alpha\eta^2(\Lambda_1-\Lambda_2)\,\xi\eta\,2\alpha\gamma(\eta^2-\eta_P^2), \tag{39f}$$

with b, c, and η_P^2 defined by

$$b \equiv (\gamma+2\alpha)\,[\xi^2+\eta^2+\mu(1-\mu)], \tag{40a}$$

$$c \equiv 2\alpha\gamma[(\eta^2-\eta_P^2)(\xi^2+\eta^2)+\eta^2\mu(1-\mu)], \tag{40b}$$

$$2\alpha\gamma\eta_P^2 \equiv (1+\alpha+\beta)(1+\alpha+\beta-\gamma). \tag{40c}$$

The double root $n^2=0$ for Eq. (37) with ξ, η, μ arbitrary is physically spurious. The mathematical origin of one extraneous root is as follows. We have already remarked that the adoption of Eq. (12) rather than Eq. (2e) involves an unnecessary time differentiation. This time differentiation comes back to haunt us since the eighth and ninth columns of the matrix A are identical except for a factor β. Thus a formal solution of Eq. (36) is given by $n=0$, $\Pi=-\beta\Pi_{cr}$ with every other amplitude equal to zero. Such a static balance between the gas pressure and the cosmic-ray pressure obviously cannot be maintained in the direction parallel to the magnetic field; consequently, the extra root $n=0$ arises only because we have used the linearized form of Eq. (3) rather than the linearized form of Eq. (2e).

The origin of the second $n=0$ root is equally extraneous. For $n \neq 0$, the seventh row of the matrix Eq. (36) implies $U_z \propto n$. But if $U_z \propto n$ and $n\to 0$, the eighth row equals γ times the first row. This implies that the equation of continuity and the adopted Eq. (2d) governing the pressure variations place no restrictions on the relative variations of the gas pressure and the gas density for static disturbances. This arbitrary degree of freedom leads formally to the appearance of a "neutrally stable mode". Clearly, this "mode" has little physical basis without a detailed consideration of the thermodynamics, and we ignore it in what follows.

The physically interesting situations correspond to $\Delta(n)=0$. In the expression for $\Delta(n)$, however, the retention of certain terms are clearly incompatible with the weak-shear approximation $\eta(\Lambda_1-\Lambda_2)/\xi n \to 0$; for example Cn^4 is obviously much smaller than the term $(\gamma+2\alpha)\,\xi^2 n^5$ in Bn^5. The systematic removal of these small terms, which are properly considered only when the $N=1$ contributions from the series in Eq. (35) are also considered, results in the simplified dispersion relation $\Delta(n) \simeq n\Delta_0(n)=0$ where $\Delta_0(n)$ is the polynomial, $\Delta_0(n)=n^6+Bn^4+Dn^2+E_0 n+F_0$, with B and D given by equations (39a) and (39c), but with E_0 and F_0 given by

$$E_0 = 2\alpha(1+\alpha+\beta)\,\Lambda_1\xi\eta(1-2\mu), \qquad F_0 = 2\alpha\eta^2 c. \tag{41}$$

In a consistent weak-shear approximation, therefore, the shear parameter $(\Lambda_1-\Lambda_2)$ does not enter anywhere in the dispersion relation.

The new marginally stable "mode" implied by $n\Delta_0(n)=0$ is again artificial. To discover its origin requires a separate discussion. Clearly, $n\to 0$ can be consistent with the weak shear assumption $\eta(\Lambda_1-\Lambda_2)/\xi n\to 0$ for arbitrary ξ, η, μ only for $\Lambda_1=\Lambda_2$ (uniform rotation). For $\Lambda_1=\Lambda_2$ with ξ, η, μ arbitrary, the equation of continuity and the x, z components of the field-freezing equations demand $\boldsymbol{U}=0$. Force balance in the three directions now requires

$$2i\alpha\eta\mathscr{B}_x - 2i\alpha\xi\mathscr{B}_y - i\xi(\Pi+\beta\Pi_{cr}) = 0, \tag{42a}$$

$$\alpha\mathscr{B}_z - i\eta(\Pi+\beta\Pi_{cr}) = 0, \tag{42b}$$

$$(1+\alpha+\beta)\,S + 2\alpha(\mu-1)\,\mathscr{B}_y + 2i\alpha\eta\mathscr{B}_z + (\mu-1)(\Pi+\beta\Pi_{cr}) = 0. \tag{42c}$$

To remove the artificial "degeneracy" introduced by the two extraneous $n=0$ roots, we note that Eq. (2e) requires

$$-\beta(\mathscr{B}_z + i\eta\Pi_{cr}) = 0, \tag{43}$$

whereas, for $\boldsymbol{u}=0$, an integration of Eq. (1) yields upon linearization,

$$\Pi - \gamma S = 0. \tag{44}$$

Thus, the appearance of the new $n=0$ root can be associated with the y-component of the field-freezing Eq. (20f) being trivially satisfied for $\boldsymbol{u}=0$ and $\partial b_y/\partial\tau=0$ when $\Lambda_1=\Lambda_2$. But in circumstances when all three components of the field-freezing equations are trivially satisfied, we must make certain that the perturbations are consistent with the constraint $\nabla\cdot\boldsymbol{B}=0$, i.e., with

$$-i\xi\mathscr{B}_x - i\eta\mathscr{B}_y + (\mu-1/2)\,\mathscr{B}_z = 0, \tag{45}$$

for the present problem. The six Eqs. (42)–(45) generally only allow the trivial solution, $S=\mathscr{B}_x=\mathscr{B}_y=\mathscr{B}_z=\Pi=\Pi_{cr}=0$.

The modes of physical interest correspond, therefore, to

$$\Delta_0(n) = n^6 + Bn^4 + Dn^2 + E_0 n + F_0 = 0. \tag{46}$$

Within the context of the weak-shear approximation, an arbitrary disturbance can be synthesized by a judicious superposition of the six distinct types (and two parities) of normal-mode solutions if they form a complete set. To avoid non-physical static components of the disturbance, the initial values of the disturbance should be consistent with the three constraints (43), (44), (45).

Among the six roots in Eq. (46) are the roots corresponding to Parker's mode. To help our later sorting out of the various modes, we find it useful first to consider two limiting forms taken by Eq. (46). To provide physical interpretations, we remark that the terms proportional to $\Lambda_1\Lambda_2$ and to $(1+\alpha+\beta)$ in the coefficients B, D, E_0, and F_0 represent, respectively, the interactions associated with rotation and with the vertical gravity.

b) Two Limiting Wavenumber Regimes

i) No Propagation Parallel to the Galactic Plane

For ξ^2, $\eta^2 \to 0$ but ζ finite, the dispersion relation (46) becomes

$$n^2\{n^4+[\Lambda_1\Lambda_2+(\gamma+2\alpha)\,\mu(1-\mu)]\,n^2 + \Lambda_1\Lambda_2(\gamma+2\alpha)\,\mu(1-\mu)\}=0\,. \tag{47}$$

Two roots are zero (corresponding, as it turns out, to Parker's mode); the nonstatic part of the dispersion relation can be factored to give for n^2:

$$n^2=-\Lambda_1\Lambda_2 \quad \text{or} \quad n^2=-(\gamma+2\alpha)\,\mu(1-\mu)\,. \tag{48}$$

The stable pair $n^2=-\Lambda_1\Lambda_2=-(\varkappa H/a)^2$. independent of μ, correspond to non-propagating Lindblad oscillations (Lindblad, 1959). The coupling of these oscillations onto self-gravitating disturbances provide a strong stabilizing influence for the Jeans' instability in a rotating disk (Toomre, 1964; see also Goldreich and Lynden-Bell, 1965a). We shall find that they are not nearly as effective for helping to stabilize Parker's mode.

The other mode in Eq. (48) depends on μ through the product $\mu(1-\mu)$. Since the solution for n cannot depend on the sign of $\zeta=-\mathrm{Im}(\mu)$, a little algebra shows that μ must have the form

$$\mu=1/2-i\zeta \tag{49}$$

which lies on the extreme end of the requirement $k=\mathrm{Re}(\mu)\leqq 1/2$. The dispersion relation for waves which propagate exclusively in the vertical direction now reads

$$n^2=-(\gamma+2\alpha)\,(\zeta^2+1/4)\,. \tag{50}$$

Clearly, the $\zeta^2\to\infty$ limit of the above equation gives the usual dispersion relation for magnetosonic waves; the additive term 1/4 is the modification introduced by the vertical stratification. Note that the amplitude of the fluid velocity perturbations, $a\boldsymbol{u}$, associated with the propagating components of the standing wave vary with z as $\exp(kz')=\exp(z/2H)$, whereas the amplitudes of the magnetic field perturbations, $B_0\boldsymbol{b}$, do not change with z. We can understand this result physically in a very simple way – at least, for wavelengths which are much shorter than the scale height H. For such short waves, the wave energy averaged over one wavelength is conserved during propagation. Since this wave energy is proportional to both $B_0^2|\boldsymbol{b}|^2$ and $\varrho_0|\boldsymbol{u}|^2$ and since $\varrho_0(z)\propto B_0^2(z)\propto\exp(-z/H)$, it is only "natural" that $\boldsymbol{u}$ and $\boldsymbol{b}$ should vary as $\exp(z/2H)$.

We disagree with Lerche and Parker's (1967) feeling that the aspect of normal modes having finite energy density at infinity causes conceptual difficulties inasmuch as it generally takes an *infinite time* for any normal mode to be realized from an initially localized disturbance. Thus, in our formulation of the initial-value problem (§ IIIa), only the long-term development of the system would be characterized by exponential growth of the mode with the largest growth rate. In practice, of course, it is questionable whether such a state would be reached before nonlinear effects – such as the development of shocks from the steepening of outwardly propagating waves – would intervene. Such shocks may be of practical importance, for instance, for the heating of a hypothetical halo gas. In any case, in the context of the linear theory and the specific model being considered, we would assert that the normal-mode analysis of Parker (1966, 1967b) for class A disturbances is rigorously correct (cf. Lerche and Parker, 1967).

ii) Short High-frequency Waves

For $\mu=1/2-i\zeta$, and ξ^2, η^2, ζ^2, $n^2\to\infty$, the dispersion relation (46) becomes

$$\begin{aligned} &n^6+[(\gamma+2\alpha)(\xi^2+\eta^2+\zeta^2)+2\alpha\eta^2]\,n^4\\ &\quad+4\alpha(\gamma+\alpha)\,\eta^2(\xi^2+\eta^2+\zeta^2)\,n^2\\ &\quad+4\alpha^2\gamma\eta^4(\xi^2+\eta^2+\zeta^2)=0\,. \end{aligned} \tag{51}$$

Note that the dynamical effects of differential rotation, the vertical gravity, and the cosmic-ray pressure drop out for wavelengths which are much shorter than the vertical scale height in all three dimensions. The above dispersion relation can be factored, and the resultant roots for n^2 are given by

$$\begin{aligned} n^2&=-2\alpha\eta^2\,,\\ n^2&=-\tfrac{1}{2}(\xi^2+\eta^2+\zeta^2)\{(\gamma+2\alpha)\\ &\quad\pm[(\gamma+2\alpha)^2-8\alpha\gamma\cos^2\psi]^{1/2}\}\,, \end{aligned} \tag{52}$$

where $\cos^2\psi\equiv\eta^2/(\xi^2+\eta^2+\zeta^2)$ and ψ is the angle between the propagation vector (ξ,η,ζ) and the direction of the unperturbed magnetic field.

The first root for n^2 corresponds to Alfvén waves with oblique wave fronts propagating along the field lines. The other two roots correspond to the conventional fast and slow modes of hydromagnetic theory [5]).

V. Parker's Mode of Instability

For moderate wavelengths in the galactic plane, the vertical gravity of the Galaxy (as well as the cosmic-ray pressure) constitute important modifying influences – so much so as to destabilize the system (Parker, 1966, 1967b). To follow the nature of this instability, we consider the problem in two separate stages.

[5]) The role of the finite inertia of the cosmic-ray gas, neglected here, is to replace the fast mode by the suprathermal mode for wave propagation nearly exactly perpendicular to the field lines (Parker, 1965). This subtle effect is of no consequence for the present problem inasmuch as we shall find at the short wavelength limit that Parker's mode corresponds to the *slow* mode of hydromagnetics (§ Va, see also Ames, 1973).

a) No Rotation and No Shear

The case $\Lambda_1 = \Lambda_2 = 0$ corresponds to Parker's problem. Various aspects of this problem have been treated by Parker (1966, 1967b); these earlier results can be recovered by considering various limits of the full dispersion relation $\Delta_0(n) = 0$ which now reads

$$n^6 + B_{00} n^4 + D_{00} n^2 + F_0 = 0 \tag{53}$$

with $B_{00} = b + 2\alpha\eta^2$, $D_{00} = 2\alpha\eta^2 b + c + 2\alpha(1+\alpha+\beta)\xi^2$, $F_0 = 2\alpha\eta^2 c$, and with b and c given by Eqs. (40a) and (40b).

To be able to satisfy the vertical boundary conditions, we can easily verify that we again require $\mu = 1/2 - i\zeta$. For this choice, $\mu(1-\mu) = \zeta^2 + 1/4$ and $b > 0$; hence a sufficient criterion for instability is $c < 0$, i.e.,

$$2\alpha\gamma[(\eta^2 - \eta_P^2)(\xi^2 + \eta^2) + \eta^2(\zeta^2 + 1/4)] < 0, \tag{54}$$

since there would then exist one root of Eq. (53) with $n^2 > 0$. We shall see in § Vc that the condition (54) is also generally a necessary criterion for the instability of Parker's mode.

i) The Limit $\alpha = \beta = 0$

For $\alpha = \beta = 0$ (whenever we set $\alpha = 0$ and not just α very small, we must also set $\beta = 0$ because the cosmic-rays are decoupled from the thermal gas in every direction in the absence of an interstellar magnetic field), the instability criterion (54) becomes $-2\alpha\gamma\eta_P^2 = (\gamma - 1) < 0$; i.e., instability occurs for $\gamma < 1$. This criterion is the analogue of Schwarzschild's criterion for *convective* instability in stars. The development of the instability depends purely on considerations of the excess buoyancy of a bubble of gas which is displaced from its equilibrium position and whose internal pressure varies with density as $P \propto \varrho^\gamma$ when the follow the motion of the bubble. In a star, the bubble is assumed to behave adiabatically so instability arises if the ambient medium is heated (from below) so vigorously that the temperature gradient exceeds in absolute amount the adiabatic temperature gradient. In the interstellar medium, we have assumed some source of *volume* heating of the gas which keeps the ambient medium at a uniform temperature (or state of "turbulence" if one prefers to identify *a* with the random cloud motions). Instability sets in here if the rising bubble (descending bubble) heats up (cools down) as it expands (contracts) under the lower (higher) ambient pressure. The origin of the instability is thermal because of the radiative character of the interstellar gas (Savedoff and Spitzer, 1950), but the instability is quite distinct from the condensation (and evaporation) process investigated comprehensively in the paper of Field (1965).

Because this point will be important for our later interpretations, we shall pursue the matter a little further. For $\alpha = \beta = 0$, the dispersion relation (53) degenerates into a quadratic equation to solve for n^2. This has the usual solution

$$n^2 = \tfrac{1}{2}\{-\gamma(\xi^2 + \eta^2 + \zeta^2 + 1/4) \pm [\gamma^2(\xi^2 + \eta^2 + \zeta^2 + 1/4)^2 + 4(1-\gamma)(\xi^2 + \eta^2)]^{1/2}\}. \tag{55}$$

For $\gamma < 1$, the unstable solution corresponds to the choice of the plus sign. Note that instability occurs for all wavelengths (as long as propagation does not occur exclusively in the vertical direction). For fixed $(\xi^2 + \eta^2) \neq 0$, the growth rate is maximized for $\zeta = 0$, i.e., for infinite vertical wavelength. For ζ set formally equal to zero, the largest growth rate is achieved for $(\xi^2 + \eta^2) \to \infty$. In the double limit $\zeta \to 0$ and $(\xi^2 + \eta^2) \to \infty$, we find for the unstable mode,

$$n^2 = (1-\gamma)/\gamma, \tag{56}$$

and the corresponding eigenvector components obtained from Eq. (36) are

$$S : U_x : U_y : U_z : \Pi = -\frac{\gamma(\xi^2+\eta^2)}{(1-\gamma/2)} : i\left(\frac{\gamma}{1-\gamma}\right)^{1/2}\xi : i\left(\frac{\gamma}{1-\gamma}\right)^{1/2} \cdot \eta : \frac{\gamma^{3/2}(\xi^2+\eta^2)}{(1-\gamma)^{1/2}(1-\gamma/2)} : 1. \tag{57}$$

Thus, the fastest growing disturbance corresponds to rising columns of rarified (warm) material and descending columns of dense (cold) material with the fluid motions predominantly in the vertical direction and with the pressure variations in adjacent columns relatively small in comparison with the density variations. We might expect the nonlinear resolution of these instabilities which have very short scales in the (x, y) directions to result in vertical convection cells; however, we shall see that these convective motions are suppressed by a strong magnetic field. Hence, it is also of some interest to investigate the nature of the long wavelength disturbances.

For waves which are very long in all three directions, i.e., for $\xi^2 + \eta^2 + \zeta^2 \to 0$, the dispersion relation (55) with the choice of the plus sign becomes

$$n^2 = \frac{4}{\gamma}(1-\gamma)(\xi^2 + \eta^2). \tag{58}$$

Comparison with the corresponding formula (56) for the short waves suggests that the growth rate must remain fairly constant from $\xi^2 + \eta^2 = \infty$ to $\xi^2 + \eta^2 \gtrsim 1/4$, i.e., the spectrum of rapidly growing waves is very broad – the overall convection would probably become "turbulent". The eigenvector components for the long waves are

$$S : U_x : U_y : U_z : \Pi = \frac{1}{2} : \frac{i}{2}\left(\frac{\gamma}{1-\gamma}\right)^{1/2} \frac{\xi}{(\xi^2+\eta^2)^{1/2}} : \frac{i}{2}\left(\frac{\gamma}{1-\gamma}\right)^{1/2} \cdot \frac{\eta}{(\xi^2+\eta^2)^{1/2}} : \frac{(2-\gamma)(\xi^2+\eta^2)^{1/2}}{\gamma^{1/2}(1-\gamma)^{1/2}} : 1. \tag{59}$$

Thus, waves which are much longer than the vertical scale height have associated fluid motions which are primarily confined to the direction parallel to the galactic plane. Moreover, for a "stiff" pressure response, $\gamma > 1$, these long waves are stable (as would be the short waves) and have phase and group velocities given by

$$2a[(\gamma-1)/\gamma]^{1/2} = 2\{[(\gamma-1)/\gamma]\, gH\}^{1/2}, \tag{60}$$

which bears some resemblance to the usual expression for (nondispersive) shallow water waves. In the stable regime, $\gamma > 1$, $\alpha = \beta = 0$, therefore, the long wavelength counterpart of Parker's mode corresponds to gravity waves.

The marginal stability of isothermal disturbances, $\gamma = 1$, in an isothermal atmosphere is, of course, a very old result (see, e.g., Chapter X of Lamb, 1916). The modern interest in the problem rests with $\alpha \neq 0$, $\beta \neq 0$.

ii) The Limit $\xi^2 \to 0$

For $\alpha \neq 0$, $\beta \neq 0$, the dispersion relation (53) is a cubic in n^2 whose general solution we shall write down in subsection (iii). For the present, to gain insight into the physics of the problem, let us recall the special case considered explicitly by Parker (1966), $\xi^2 = 0$. For $\xi^2 = 0$, but α, $\gamma > 0$, the instability criterion (54) implies

$$\begin{aligned} \eta^2 + \zeta^2 &< \eta_P^2 - 1/4 \\ &= (1+\alpha+\beta)(1+\alpha+\beta-\gamma)/2\alpha\gamma - 1/4, \end{aligned} \tag{61}$$

i.e., waves which are sufficiently long in the plane of the magnetic field and the vertical gravity are unstable if $\eta_P^2 > 1/4$ – this is Parker's criterion. Note that the magnetic field is actually a *stabilizing* influence for the *short* waves since arbitrarily short waves (leading to vertical convection) would be unstable for $\gamma < 1 + \beta$ if $\alpha \to 0$. Short waves are stabilized, for finite gas pressure, because the tension in the magnetic field resists "kinking" on too short a scale. Large magnetic fields are, of course, destabilizing for the longer wavelengths for the reasons given by Parker, namely, because of the development of a magnetic Rayleigh-Taylor instability. In particular, if the thermal gas is taken to be cold, $a \to 0$ (i.e., $\alpha = B_0^2/8\pi\varrho_0 a^2 \to \infty$ with B_0^2 remaining finite so that $H = (1+\alpha+\beta)a^2/g$ remains finite), as in some of the models of Parker and Lerche, arbitrarily short waves are again unstable by Parker's criterion.

The important point is that the instability which sets in when the magnetic field is very weak ($\alpha \ll 1$ with $\gamma < 1$) is very different in nature from the instability which sets in when the magnetic field is very strong (relative to the gas pressure). The characteristic time for the instability to develop in all cases is of the order of the free-fall time over one scale height, $(2H/g)^{1/2}$ $\sim \omega_z^{-1}$ where ω_z is the z-oscillation frequency ($\omega_z^{-1} \sim 10^7$ years in the solar neighborhood).

Note that instability may occur even for $\gamma > 1$ and $\alpha \ll 1$ if β is sufficiently large. Such a situation corresponds to the cosmic-ray gas trying to punch "holes" in the weak magnetic field to escape from the Galaxy. It may seem strange at first sight that this cosmic-ray instability should also develop on a time scale ω_z^{-1}, but the rate-limiting process is the time required for the thermal gas to move aside as the cosmic-rays try to stream past them in the vertical direction.

For $\xi^2 = 0$, the dispersion relation (53) can be factored

$$(n^2 + 2\alpha\eta^2)(n^4 + bn^2 + c) = 0. \tag{62}$$

so that Alfvén waves transverse in the x-direction decouple from the problem. The square of the growth rate of Parker's mode is now given by

$$n^2 = \tfrac{1}{2}[-b + (b^2 - 4c)^{1/2}] \tag{63}$$

where we have chosen the plus sign in front of the radical to select the root with n^2 positive when c is negative. We easily verify that the above root corresponds to the slow mode of hydromagnetics in the limit $(\eta^2 + \zeta^2) \to \infty$. For $(\eta^2 + \zeta^2) < \eta_P^2 - 1/4$, we have instability, and a simple differentiation recovers Parker's result that maximum growth rates with fixed η are obtained for $\zeta = 0$, i.e., again for infinite vertical wavelength.

For $\xi^2 = \zeta^2 = 0$, the eigenvector is characterized by the relative components ($U_x = \mathscr{B}_x = \mathscr{B}_y = 0$)

$$\begin{aligned} S &: U_y : U_z : \mathscr{B}_z : \Pi : \Pi_{cr} \\ &= \Delta_1/(n^2+\gamma\eta^2) : i\Delta_2/(n^2+\gamma\eta^2) : n : -i\eta : \Delta_3/ \\ &\quad (n^2+\gamma\eta^2) : 1\,. \end{aligned} \tag{64}$$

where

$$\begin{aligned} \Delta_1 &= n^2/2 - (1+\alpha+\beta-\gamma)\eta^2, \\ \Delta_2 &= \eta n(1+\alpha+\beta-\gamma/2), \\ \Delta_3 &= n^2(1-\gamma/2) - \gamma(\alpha+\beta)\eta^2\,. \end{aligned} \tag{65}$$

(Note that the normalization to unity for Π_{cr} does not prevent us from ignoring, if we choose, the effect of the cosmic-ray pressure since the total perturbation amplitude of the cosmic-ray pressure is $P_{cr_0} p_{cr} \propto \beta\Pi_{cr}$ and we can always set $\beta = 0$. In particular, we easily check that S, U_y, U_z, and Π have the same ratios as given in Eq. (57) if we set $\alpha = \beta = 0$ and $\xi^2 = \zeta^2 = 0$.) Numerical applications will be considered in subsection (c) and in the following paper; for the present, we continue with the formal derivation of results.

iii) The General Case

An arbitrary initial disturbance would not only produce waves with $\xi^2 = 0$. To discuss the general case, we need to solve the cubic Eq. (53). The standard procedure involves substituting $n^2 = u - B_{00}/3$ to obtain a cubic equation in u without a quadratic term, $u^3 - M_0 u + N_0 = 0$, where

$$M_0 = \tfrac{1}{3}B_{00}^2 - D_{00}, \quad N_0 = \tfrac{2}{27}B_{00}^3 - \tfrac{1}{3}B_{00}D_{00} + F_0. \quad (66)$$

We then substitute $u = v + M_0/3v$ and multiply through by v^3 to obtain a quadratic equation in v^3, $v^6 + N_0 v^3 + (M_0/3)^3 = 0$. The latter has the usual solutions, $v^3 = -N_0/2 \pm \{(N_0/2)^2 - (M_0/3)^3\}^{1/2}$. For n^2 to have three real solutions it is necessary and sufficient that $(M_0/3)^3 \geqq (N_0/2)^2$, but we have already seen that n^2 has three real roots when $\xi^2 = 0$. Therefore, we define the (possibly complex) angle ϕ by

$$\cos\phi = \frac{-N_0/2}{(M_0/3)^{3/2}}, \quad (67)$$

where ϕ is considered to be an analytic function of (ξ^2, η^2, ζ^2) which is real at least on the plane $\xi^2 = 0$, and where we fix its phase by requiring

$$\phi = \pi \quad \text{at} \quad \xi^2 = \eta^2 = 0. \quad (68)$$

The roots for v^3 can now be expressed as $v^3 = (M_0/3)^{3/2} \cdot e^{\pm i(\phi + l2\pi)}$ with $l = -1, 0, +1$. The three independent solutions for $n^2 = -B_{00}/3 + v + M_0/3v$ become

$$n^2 = -\frac{B_{00}}{3} + 2\left(\frac{M_0}{3}\right)^{1/2} \cos[(\phi + l2\pi)/3], \quad l = -1, 0, +1, \quad (69)$$

where we take $(M_0/3)^{1/2}$ to be positive on the plane $\xi^2 = 0$. Note that the $l = +1$ root gives $n^2 = -(\gamma + 2\alpha) \cdot (\zeta^2 + 1/4)$ for $\xi^2 = \eta^2 = 0$, whereas the $l = -1$ and the $l = 0$ roots yield $n^2 = 0$. Thus, the $l = +1$ root corresponds to the fast mode of hydromagnetics in the short wavelength limit. To see which of the roots, $l = -1$ or $l = 0$, corresponds to Parker's mode, we consider values of ξ^2, η^2, ζ^2 which satisfy $F_0 = 0$, the condition of marginal stability. For $F_0 = 0$, $N_0 = (B_{00}/3) M_0 - (B_{00}/3)^2$, and the identity $\cos\phi = \cos(\phi/3)\{4\cos^2(\phi/3) - 3\}$ allows the solution $\cos(\phi/3) = (B_{00}/6)(M_0/3)^{-1/2}$. *Thus, Parker's mode must correspond to the $l = 0$ solution.*

iv) The Limit $\xi^2 \to \infty$

Another important limit is the limit $\xi^2 \to \infty$ (Parker, 1967b). For $\xi^2 \to \infty$, but η^2 and ζ^2 finite, B_{00}, D_{00}, and F_0 are all of order ξ^2,

$$B_{00} \simeq (\gamma + 2\alpha)\xi^2, \quad (70a)$$

$$D_{00} \simeq [2\alpha\eta^2(\gamma + 2\alpha) + 2\alpha\gamma(\eta^2 - \eta_P^2) + 2\alpha(1 + \alpha + \beta)]\xi^2, \quad (70b)$$

$$F_0 \simeq 2\alpha\eta^2 \cdot 2\alpha\gamma(\eta^2 - \eta_P^2)\xi^2. \quad (70c)$$

Since $\cos\phi \to -1$ as $\xi^2 \to \infty$, it is necessary to carry out an expansion to find all the roots. Without reproducing all the details, we find

$$\cos\phi \simeq -1 + \frac{27}{8}\left(\frac{D_{00}}{B_{00}^2}\right)^2 - \frac{27}{2}\frac{F_0}{B_{00}^3}$$
$$\Rightarrow \phi \simeq \pi - \left[\frac{27}{4}\left(\frac{D_{00}}{B_{00}^2}\right)^2 - 27\frac{F_0}{B_{00}^3}\right]^{1/2}. \quad (71)$$

Thus, since $(M_0/3)^{1/2} \simeq B_{00}/3 - D_{00}/2B_{00}$, we obtain the three roots

$$l = -1: \; n^2 \simeq \frac{1}{2B_{00}}\{-D_{00} - [D_{00}^2 - 4B_{00}F_0]^{1/2}\}, \quad (72a)$$

$$l = 0: \; n^2 \simeq \frac{1}{2B_{00}}\{-D_{00} + [D_{00}^2 - 4B_{00}F_0]^{1/2}\}, \quad (72b)$$

$$l = +1: \; n^2 \simeq -B_{00} \simeq -(\gamma + 2\alpha)\xi^2. \quad (72c)$$

The $l = -1$ and the $l = 0$ roots are independent of ξ^2 (and ζ^2 for that matter) and could, of course, have been obtained much more directly by taking the $\xi^2 \to \infty$ limit of Eq. (53) under the assumption that n^2 remains finite.

For $\xi^2 \to \infty$, the necessary and sufficient criterion for instability is $F_0 < 0$, i.e.,

$$\eta^2 < \eta_P^2 = (1 + \alpha + \beta)(1 + \alpha + \beta - \gamma)/2\alpha\gamma \quad (73)$$

independent of ζ^2. Comparison with the instability criterion (61), appropriate for $\xi^2 = 0$, shows that the above conditions is a less restrictive requirement on η^2. The physical advantage of the additional "room for expansion" gained by having $\xi^2 \to \infty$, i.e., by having alternate layers of "valleys" and "mountains", has already been explained by Parker (1967b). Our own feeling is that this slight advantage, *taken by itself*, provides only a marginal indication for the inclination of the thermal gas to "fragment" on very short scales in the direction perpendicular to the plane defined by the vertical gravity and the magnetic field. As we shall see in subsection (c), and as has already been discussed by Parker (1967b) for the case $\beta = 0$, the maximum growth rates for ξ^2 small are quite comparable with those for ξ^2 large (for fixed values of α, β, γ); whereas instability generally requires η^2 to be on the order of unity. The practical development of the instability, in the absence of other considerations (e.g., differential rotation, non-linear "cascade") may be determined more by initial conditions than by anything else, and a general initial disturbance would not especially favor $\xi^2 \gg \eta^2$. An important exception may exist concerning this last point, and it provides one of the motivations for the following paper.

To complete the discussion of this subsection, we record below the eigenvector components for the limit $\xi^2 \to \infty$:

$$S : -i\xi U_x : U_y : U_z : -i\xi \mathscr{B}_x : \mathscr{B}_y : \mathscr{B}_z : \Pi : \Pi_{cr}$$
$$= (\pi_0 + \gamma - 1)/\gamma : in\zeta : i\eta(\pi_0 + \alpha + \beta)/n : n : \eta\zeta : -(\pi_0 + \beta)/2\alpha : -i\eta : \pi_0 : 1, \quad (74)$$

where π_0 is defined to be

$$\pi_0 = [2\alpha n^2 - \gamma(\alpha+\beta)(n^2+2\alpha\eta^2)]/ [(\gamma+2\alpha)n^2+2\alpha\gamma\eta^2]. \tag{75}$$

Again, we defer numerical examples to subsection (c).

b) *The Role of Rotation and Shear*

In the presence of finite differential rotation but within the context of the weak-shear approximation $\eta(\Lambda_1-\Lambda_2)/\xi n \to 0$, the dispersion relation (46) is strictly valid for only two cases of physical interest, (i) the case $\Lambda_1=\Lambda_2\equiv\Lambda$ (uniform rotation), and (ii) the limit $\xi^2\to\infty$. For the case of uniform rotation, the condition that the solution for n does not depend on the sign of $\zeta=-\mathrm{Im}(\mu)$ leads to the requirement,

$$\zeta = 0, \tag{76a}$$

or

$$(1-2k) = \frac{4\alpha(1+\alpha+\beta)\Lambda\xi\eta n}{(\gamma+2\alpha)n^4+[(\gamma+2\alpha)\Lambda^2+4\alpha(\gamma+\alpha)\eta^2]n^2+4\alpha^2\gamma\eta^4}. \tag{76b}$$

Thus, for $\zeta\neq 0$, $k\equiv\mathrm{Re}(\mu)$ can be real only if n is real, or if $\alpha\Lambda\xi\eta=0$, or if $n^4\to\infty$. This peculiar combination of requirements on the possible normal mode solutions is vaguely reminiscent of the Taylor-Proudman theorem (see, e.g., Chandrasekhar, 1961); however, we shall discuss here only the case $\xi^2=0$ for which the physical role of uniform rotation is quite cleanly illustrated. (It is easily verified that the substitution of k given by Eq. (76b) into Eq. (46) results in a dispersion relation connecting n^2 with the *squares* of the wavenumbers ξ, η, and ζ.)

i) The Case $\Lambda_1=\Lambda_2\equiv\Lambda\neq 0$ and $\xi^2=0$

For $\Lambda_1=\Lambda_2\equiv\Lambda\neq 0$ and $\xi^2=0$, Eq. (46) becomes, with $\mu(1-\mu)=\zeta^2+1/4$,

$$n^6+Bn^4+Dn^2+F_0=0, \tag{77}$$

where $B=\Lambda^2+b+2\alpha\eta^2$, $D=\Lambda^2\{2\alpha\eta^2+(\gamma+2\alpha)\cdot(\zeta^2+1/4)\}+2\alpha\eta^2 b+c$, $F_0=2\alpha\eta^2 c$, with $b=(\gamma+2\alpha)\cdot(\eta^2+\zeta^2+1/4)$, $c=2\alpha\gamma\eta^2\{(\eta^2-\eta_P^2)+\zeta^2+1/4\}$. The cubic Eq. (77) in n^2 can be solved as before to yield

$$n^2 = -\frac{B}{3}+2\left(\frac{M}{3}\right)^{1/2}\cos[(\phi+l2\pi)/3], \quad l=-1,0,+1, \tag{78}$$

where $M=B^2/3-D$, $N=2B^3/27-BD/3+F_0$, $\cos\phi=(-N/2)(M/3)^{-3/2}$, with the phase of ϕ fixed by requiring $0\leqq\phi\leqq\pi$ for $\eta^2=0$. In the limit $\eta^2\to 0$ the three roots become

$$l=-1:\ n^2=-\tfrac{1}{2}[\Lambda^2+(\gamma+2\alpha)(\zeta^2+1/4)] +\tfrac{1}{2}|\Lambda^2-(\gamma+2\alpha)(\zeta^2+1/4)|, \tag{79a}$$

$$l=0:\quad n^2=0, \tag{79b}$$

$$l=+1:\ n^2=-\tfrac{1}{2}[\Lambda^2+(\gamma+2\alpha)(\zeta^2+1/4)] -\tfrac{1}{2}|\Lambda^2-(\gamma+2\alpha)(\zeta^2+1/4)|, \tag{79c}$$

which are the three roots, 0, $-\Lambda_1\Lambda_2$, and $-(\gamma+2\alpha)\cdot(\zeta^2+1/4)$ known from our analysis of § IVb(i). For $\eta^2\neq 0$, we can again identify the $l=0$ root with Parker's mode.

It is a straightforward matter to prove that $M>0$ for all real η and ζ. This implies that ϕ can either be real or pure imaginary, and, therefore, that the transition from stability to instability for Parker's mode must occur by an exchange of stabilities since the appropriate root for n^2 is always real. Thus, the marginally stable state for Parker's mode can be found by examining the form taken by the dispersion relation when $n^2=0$. Equation (77) shows then $F_0=0$ to give the locus of the marginally stable state, and instability corresponds to $F_0<0$, i.e., *to the inequality* (61).

This is a rather remarkable result because it implies that Parker's criterion (61) for the onset of instability of modes with $\xi^2=0$ is not affected by rotation if there is no shear (when the rotation axis coincides with the vertical gravity). The physical explanation is as follows. The effect of rotation on the perturbations enters only in the "Coriolis force". On the margin of stability, i.e., for $n=0$, the matrix Eq. (36) demands $\boldsymbol{U}=0$. But rotational coupling through the Coriolis force does not enter without fluid motions; therefore, in the absence of shear, it is always possible for the gas to drain down the field lines (along constant "radii") so *slowly* that the effects of rotation do not manifest themselves in the corotating frame. This explains why the Lindblad oscillations (the root $n^2=-\Lambda_1\Lambda_2$ for $\zeta^2+\eta^2\to 0$) are not coupled with Parker's mode, the $l=0$ root, but with the $l=\pm 1$ mode, depending on whether $\Lambda_1\Lambda_2 \gtrless (\gamma+2\alpha)(\zeta^2+1/4)$. This situation is in marked contrast with the corresponding situation for the Jeans' instability in rotating, self-gravitating, disks where it is known (Toomre, 1964, see also Goldreich and Lynden-Bell, 1965a) that rotational coupling provides a strong stabilizing influence for the long waves.

Although uniform rotation cannot prevent the *onset* of instability, it can materially affect the growth rate of any unstable disturbance with a finite development time. Indeed, a simple differentiation of the dispersion relation shows that the growth rate is maximized if the rotation parameter is zero. Quantitative measures of the stabilizing influence of rotation are given in subsection (c).

ii) The Limit $\xi^2 \to \infty$

For $\xi^2 \to \infty$, η^2 finite, the weak-shear approximation $|\eta(\Lambda_1 - \Lambda_2)/\xi n| \ll 1$ will be satisfied for $\Lambda_1 \neq \Lambda_2 \neq 0$ as long as n is nonzero. In the limit $\xi^2 \to \infty$, the dispersion relation (46) is, therefore, rigorously accurate and takes the form

$$n^6 + Bn^4 + Dn^2 + F_0 = 0,$$

where $B = (\gamma + 2\alpha)\xi^2$, $D = \{2\alpha\eta^2(\gamma + 2\alpha) + 2\alpha\gamma(\eta^2 - \eta_P^2) + 2\alpha(1 + \alpha + \beta)\}\xi^2$, and $F_0 = 2\alpha\eta^2 \cdot 2\alpha\gamma(\eta^2 - \eta_P^2)\xi^2$, and all dependence on Λ_1 and Λ_2 dropout. The three roots for n^2 are now *exactly* the same as given in § Va(iv); hence, we shall not repeat the results here.

The important point is that both rotation and shear are ineffective for helping to stabilize Parker's mode if the characteristic scale in the radial direction is very short. The nonobvious part of this conclusion is that even the unstable disturbances with a finite development time and finite fluid motions are not hampered by the effects of the Coriolos force when the scale in the radial direction is very short.

c) *Numerical Examples*

Tabulated in Table 1 for a variety of values of the squares of the horizontal wavenumbers, ξ^2 and η^2, are the squares of the growth rates for Parker's mode when $\alpha = \beta = \gamma = 1$, $\Lambda_1 = \Lambda_2 = 0$, with $\zeta^2 = 0$. Positive values of n^2 correspond to exponential dependences on time (instability); negative values, to purely oscillatory waves (stability). (We have checked for a fairly representative sample of parameter space that the $l = \pm 1$ modes yield only stable waves.) The salient feature exhibited by Table 1 is the flatness of the eigenvalue spectrum as a function of the wavenumber ξ for given η (see also Fig. 1 of Parker (1967b) where the diffusion of the magnetic field through the gas is taken into account.) For example, the slight difference between the values for n^2 for $\xi^2 = 0$ and $\xi^2 = \infty$ when $\eta^2 = 0.8$ (corresponding to values close to the absolute maximum of the growth rate) imply 159 *e*-folding times for the purely two-dimensional Fourier components ($\xi^2 = 0$) for every 160 *e*-folding times of the Fourier components which are highly corrugated in the *x*-direction ($\xi^2 = \infty$). Since 160 *e*-folding times correspond to an increase in amplitude by a factor $\sim 10^{69}$, this difference in the growth rates can hardly be significant for the real problem because nonlinear effects are bound to come into play long before the differences in growth rates *per se* lead to a selection of the Fourier components with very large values of ξ^2. In the absence of other effects, therefore, the earlier stages of the development of the instability would be determined more by initial conditions than by anything else, in accordance with our remarks in § Va(iv). Of course, it is conceivable – and in the case when the magnetic field is very weak, perhaps, even plausible – that the eventual nonlinear state would correspond to fully developed turbulence, and that nonlinear wave-coupling would naturally lead to a "cascade" of energies to ever-shorter scales (see, e.g., Landau and Lifshitz, 1959 for a discussion of the concept of energy flow from large to small eddies in the turbulent flow of an incompressible medium); however, this possibility is well beyond the scope of the present investigation. All we can say at this point is that there exists little evidence, *within the context of the linear theory*, for expecting very short scales (say, 1–10 pc) to emerge as a dominant feature from the physical problem where the characteristic length scale is $H \sim 10^2$ pc.

The role of (uniform) rotation is to suppress waves which are very long in the horizontal directions. This effect is illustrated in Table 2 for $\xi^2 = \zeta^2 = 0$ when $\alpha = \beta = \gamma = 1$. For given $\eta^2 < \eta_P^2 - 1/4 (= 2.75$ in this

Table 1. Parker's mode the square of the growth rate n^2 for $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\Lambda_1 = \Lambda_2 = 0$ with $\zeta^2 = 0$

η^2 \ ξ^2	0	2	4	6	8	∞
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.5398	0.6188	0.6272	0.6304	0.6321	0.6376
0.4	0.7074	0.7462	0.7518	0.7541	0.7553	0.7594
0.6	0.7758	0.7927	0.7957	0.7970	0.7977	0.8000
0.8	0.7916	0.7972	0.7983	0.7988	0.7990	0.8000
1.0	0.7737	0.7744	0.7746	0.7747	0.7747	0.7749
1.2	0.7320	0.7322	0.7323	0.7323	0.7323	0.7324
1.4	0.6723	0.6751	0.6759	0.6762	0.6764	0.6772
1.6	0.5985	0.6062	0.6083	0.6093	0.6099	0.6122
1.8	0.5133	0.5274	0.5316	0.5337	0.5348	0.5394
2.0	0.4185	0.4404	0.4473	0.4506	0.4526	0.4603
2.2	0.3157	0.3464	0.3563	0.3613	0.3642	0.3759
2.4	0.2060	0.2462	0.2597	0.2665	0.2706	0.2871
2.6	0.0903	0.1405	0.1580	0.1669	0.1724	0.1945
2.8	−0.0307	0.0300	0.0519	0.0632	0.0700	0.0987
3.0	−0.1564	−0.0848	−0.0583	−0.0444	−0.0359	0.0000

Table 2. Parker's mode the square of the growth rate n^2 for $\alpha = 1$, $\beta = 1$, $\gamma = 1$ with $\xi^2 = \zeta^2 = 0$

η^2 \ Λ^2	0	1	2	3	4	5
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.5398	0.2729	0.1654	0.1155	0.0880	0.0709
0.4	0.7074	0.4688	0.3353	0.2563	0.2057	0.1712
0.6	0.7758	0.5680	0.4370	0.3509	0.2914	0.2484
0.8	0.7916	0.6117	0.4907	0.4064	0.3452	0.2993
1.0	0.7737	0.6189	0.5104	0.4317	0.3729	0.3274
1.2	0.7320	0.6000	0.5047	0.4337	0.3793	0.3364
1.4	0.6723	0.5613	0.4793	0.4169	0.3682	0.3293
1.6	0.5985	0.5069	0.4381	0.3848	0.3427	0.3085
1.8	0.5133	0.4398	0.3838	0.3399	0.3047	0.2760
2.0	0.4185	0.3621	0.3186	0.2841	0.2562	0.2331
2.2	0.3157	0.2754	0.2440	0.2189	0.1983	0.1813
2.4	0.2060	0.1810	0.1613	0.1454	0.1324	0.1214
2.6	0.0903	0.0798	0.0715	0.0648	0.0592	0.0545
2.8	−0.0307	−0.0273	−0.0246	−0.0223	−0.0205	−0.0189
3.0	−0.1564	−0.1397	−0.1263	−0.1152	−0.1060	−0.0981

example), the growth rate decreases monotonically for increasing values of $\Lambda^2=\Lambda_1\Lambda_2$. The decrease is most prominent for small η^2, but no finite amount of rotation can completely stabilize the disturbances at any $\eta^2<\eta_P^2-1/4$ for the reasons given in § Vb(i). Moreover, the band of values of n^2 which are reasonably large is narrower in the presence of rotation than in its absence – for example, for $\Lambda^2=1$, the Fourier components with values of η^2 between 0.2 and 2.2 would begin to dominate the other Fourier components after about two e-folding times if the initial spectrum of amplitudes in η was reasonably flat. This would seem to imply a tendency to form dense clumps of matter with a semi-regular spacing centered about $2\pi H$ in the direction along the magnetic field; however, a more precise statement awaits a numerical survey of the initial-value problem with the inclusion of the effects of finite shear.

To complete the discussion in this subsection, we consider the nature of the instability for various values of α, β, γ. The essential qualitative features can already be obtained by restricting our attention to the case $\Lambda_1=\Lambda_2=0$.

As our first example, we consider the case $\alpha=\beta=\gamma=1$, $\xi^2=\zeta^2=0$, and $\eta^2=1$. The square of the growth rate n^2 can be obtained from Table 1 as 0.77, and Eq. (64) yields the eigenvector components as

$$S:U_y:U_z:\mathscr{B}_z:\Pi:\Pi_{cr} = -0.91:i1.2:0.88:-i.1.0:-0.91:1\,. \tag{80}$$

Note that s and p are 180° out of phase with respect to u_z and p_{cr}, whereas u_y and b_z are 90° out of phase with respect to u_z and p_{cr}. This situation corresponds to the conventional picture of gas drainage down and along the field lines with cosmic-ray inflation in between the dense clumps.

For $\alpha=\beta=\gamma=1$, $\xi^2\to\infty$, $\zeta^2=0$, and $\eta^2=1$, n^2 can be obtained from Table 1 as 0.77, but Eq. (74) now give the eigenvector components as

$$S:U_y:U_z:\mathscr{B}_y:\mathscr{B}_z:\Pi:\Pi_{cr} = -0.92:i1.2:0.88:-0.04:-i1.0:-0.92:1\,. \tag{81}$$

Apart from the slightly nonzero value for $\mathscr{B}_y$, the above numbers are quite similar to those in Eq. (80). Although the magnetic Rayleigh-Taylor instability is operative in the case $\xi^2\to\infty$ also, we must remember that the dependence of the perturbations on $e^{-i\xi x}$ implies the presence of strong "shearing motions" across the "front" and "back" faces of the dense clumps (Parker, 1967a, b).

As a final example, we consider the case $\alpha=0.1$ (a weak magnetic field), $\beta=1$, $\gamma=5/3$, $\xi^2\to\infty$, $\zeta^2=1$, $\eta^2=1<\eta_P^2=2.73$. The square of the growth rate n^2 can be obtained from formula (72b) as 0.20, and we recover the eigenvector components from Eq. (74) as

$$S:-i\xi U_x:U_y:U_z:-i\xi\mathscr{B}_x:\mathscr{B}_y:\mathscr{B}_z:\Pi:\Pi_{cr} = -0.19:i0.44:i0.27:0.44:1.0:-0.09:-i1.0:-0.98:1\,. \tag{82}$$

The variation of the gas density is, thus, relatively smaller than in the preceeding cases. The instability exhibited in this last example is primarily a cosmic-ray instability.

VI. Discussion

As a result of his studies of the instability mechanism in the combined system of thermal gas, cosmic-ray gas, and magnetic field, Parker has concluded that the physical state of the general interstellar medium must be a complex and dynamical one. The results of the present investigation tend, if anything, to reinforce this viewpoint; the wide range of parameter and wavenumber space for which instability is found seems to warrant the application of the overused word, "turbulent", to describe the probable *present* state of the interstellar medium.

It appears likely, moreover, that when the magnetic field is strong, this "turbulent" medium can produce dense concentrations of gas by the process of the thermal gas sliding down the magnetic lines of force toward the midplane of the Galaxy (magnetic Rayleigh-Taylor instability). The effect of the differential rotation of the Galaxy is to suppress wavelengths which are very long in the horizontal directions; however, no finite amount of rotation or shear can completely stabilize Parker's mode of instability if the disturbances have very short scales in the "radial" direction. The length scale of the separation between gas clumps produced by the action of the magnetic Rayleigh-Taylor instability is characteristically of the order $2\pi H\sim 1$ kpc in the direction parallel to the magnetic field, and we believe that shear may place a more restrictive requirement on the spatial geometry in the direction perpendicular to the vertical gravity and the magnetic field. Numerical calculations of the general initial-value problem along the lines formulated in § III are needed to assess accurately the full effects of shear; however, enough of the qualitative features of the problem have emerged from our present analysis to allow us to make some preliminary remarks concerning the implications of the general processes for the interstellar medium in spiral galaxies. These speculative ideas form the basis for the paper following this one.

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Appendix

Parker (1966, 1968, 1969) has pointed out that the confinement of the interstellar gas to an observed thickness provides a method to estimate the mass density of the local interstellar gas. We believe that improved observational data warrants a reexamination of this problem.

To begin, we remark that any precise estimate of the vertical confinement problem should take into account the z-variation of the vertical gravity field. For good observational and theoretical reasons (Oort, 1965), $g(z)$ in the solar neighborhood can be approximated as a linear function of z throughout the thin layer in which the gas is observed to be confined. Magnetostatic equilibrium under the same assumptions as made in the text now leads to a gas density distribution of gaussian form

$$\varrho_0(z) = \varrho_0(0) \exp\left[-\frac{\omega_z^2 z^2}{2(1+\alpha+\beta)\,a^2}\right]. \tag{A.1}$$

where $g(z) = \omega_z^2 z$ and ω_z is the z-oscillation frequency. In the solar neighborhood, Oort's analysis corresponds to $\omega_z \simeq 90$ km s^{-1} kpc^{-1}.

Let us define the equivalent thickness $2H$ as the total thickness of a uniform slab of gas of density $\varrho_0(0)$ which leads to a given surface density. For an exponential atmosphere, H is just the equivalent isothermal scale height given by Eq. (9), but for a gaussian distribution,

$$H = \frac{a}{\omega_z}\left[\frac{\pi}{2}(1+\alpha+\beta)\right]^{1/2}. \tag{A.2}$$

The square-root dependence on the combination $(1+\alpha+\beta)$ rather than a linear dependence is important for any discussion where α and β are allowed to vary. Equation (A.2) can also be written

$$\varrho_0(0) = \frac{B_0^2(0)/8\pi + P_{cr_0}(0)}{2(\omega_z H)^2/\pi - a^2}. \tag{A.3}$$

The above equation will yield useful estimates for the gas density in the midplane (where the sun is very nearly located) as long as the denominator does not involve the subtraction of nearly equal numbers.

From the work of Radhakhrishnan *et al.* (1972), Falgarone and Lequeux (1973), and Manchester (1972), we estimate $H = 1/6$ kpc, $a = 6.4$ km s^{-1}, and $B_0(0) = 3 \times 10^{-6}$ Gauss. The cosmic-ray pressure, $P_{cr_0}(0)$, taken to be one-third the cosmic-ray energy density, $\sim 1.3 \times 10^{-12}$ erg cm^{-3} (Meyer, 1969), is about 0.43×10^{-12} dyne cm^{-2}. If we ignore the two-component aspect of the thermal gas, Eq. (A.3) together with the value $\omega_z = 90$ km s^{-1} kpc^{-1} now yields $\varrho_0(0) = 0.8 \times 10^{-24}$ gm cm^{-3}, with an uncertainty which is probably less than a factor of two. If we further adopt a contribution of 10% by number of helium atoms, we obtain the number density of hydrogen nuclei in the central plane to be about 0.4 cm^{-3}. This value is close to the observationally derived values obtained from studies of 21-cm profiles in emission and absorption (Hughes *et al.*, 1971; Radhakhrishnan *et al.*, 1972) and satellite determinations of the neutral hydrogen density based on interstellar Lyman α absorption (Savage and Jenkins, 1972). This line of reasoning seems to indicate that the contribution of molecular hydrogen to the local gas density probably does not exceed the contribution of atomic hydrogen – in accord with the estimate of Hollenbach, Werner, and Salpeter (1971) that 20–40% of the total mass of hydrogen within about 1 kpc of the sun is in the form of molecular hydrogen. In conclusion, then, a reasonable estimate for α and β in the solar neighborhood is $\alpha \simeq \beta \simeq 1$.

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