Color algebra of three quarks

S.-C. Lee

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
and The Institute for Advanced Study, Princeton, New Jersey 08540
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The color algebra with the outer product \([u,u]\)^4 = f^{ABC}(u^A_{B}v^C + v^C_{B}u^A) is studied for the case of three-quark sources. It is shown to contain two Abelian elements which annihilate the color-singlet state and a sixteen-element ideal which contains an eight-element subalgebra isomorphic to \(u(2) \oplus u(2)\). The Jacobi identity is satisfied on the whole algebra. The quantity that measures the breakdown of the Jacobi identity is calculated.

I. INTRODUCTION

In a series of papers,^1,2 Adler proposed a method of taking into account the quantization of color charges by constructing an equivalent classical theory. One basic idea is to consider the vector space generated by the quark charges under linear and certain bilinear operations. It turned out to be finite dimensional and was called the underlying algebra or color algebra. An inner product \(\langle , , \rangle\) and an outer product \([ , ,]\) can be defined on the color algebra. Two conditions have to be satisfied in order to get an underlying classical gauge theory. One is the Jacobi identity for the outer product and the other is the trace condition \(\langle u,[v,w]\rangle = 0\). In Ref. 4 I found all the outer products that satisfy the Jacobi identity. The trace condition in the form originally assumed by Adler cannot be satisfied in general for these outer products. In a recent paper,^2 Adler proposed to use the natural outer product \([u,v]^A = f^{ABC}(u^Bv^C + v^C_{B}u^A)\) and an inner product differing from \(\langle u,v\rangle\) = \(\text{Tr}(u^Av^A)\) only by rescalings within diagonal subalgebras. The trace condition is automatically satisfied while the Jacobi identity is not. As shown by Adler,^2 the trace condition is enough to guarantee the existence of a Lagrangian that gives the proper equations of motion. The outer product appears explicitly in the equations of motion and it becomes necessary to study the structure of the color algebra with respect to the outer product. For two quarks (or a quark and an antiquark), the underlying algebra is a Lie algebra. In the present work, I show that the three-quark color algebra contains two elements which commute with the whole algebra and annihilate the color-singlet state. The remaining subalgebra, denoted as \(V\), is sixteen dimensional and does not contain any proper subideal. \(V\) has a subalgebra \(V_1\) which is a Lie algebra and is isomorphic to \(u(2) \oplus u(2)\).

In the following section, I indicate how to compute outer and inner products. In Sec. III, the main results for the structure of outer products are described with brief explanations as to how they were derived. Discussion of the results is given in the final section.

II. THE INNER AND OUTER PRODUCTS

I shall follow the notations of Ref. 4. In particular, since the color algebra for \(N\) quarks with the product \((u \times v)^A = q^{ABC}u^Bv^C\) is isomorphic to the group algebra \(S^{N}_A\), we can define an inner product on \(S^{N}_A\) through the isomorphism (denoted as \(\text{IT}\)):

\[
\langle g,h \rangle = IT^{-1}(g) IT^{-1}(h) = \text{Tr}(u^Av^A)
\]

(2.1)

if \(IT(u) = g, IT(v) = h\).

Note that \(u^Av^A\) is a \(u(n)\) scalar and that under the isomorphism \(IT^{-1}\), the subalgebra \(S^2\) is mapped onto elements of the form \(u^Av^A\), where \(u^A\) is a certain (operator valued) \(u(n)\) scalar and \(v^A\) is the identity element of the product \(\times\). (Recall that \(S^2\) denotes the subgroup of \(S^{N}_A\) which leaves zero unpermuted and \(v^A\) is defined as \(e^A = \sqrt{2n} \delta^{AB}\).) This motivates the following definition:

Let \(g^2 \in S^2\) and \(IT^{-1}(g^2) = u^Av^A\), then define

\[
\text{Tr} g^2 = \text{Tr} u^Av^A.
\]

(2.2)

We want to find \(IT(u^Av^A) \in S^2\) given that \(IT(u) = g, IT(v) = h\). From the definition of the product, we have

\[
(u^Av^A)v = (2n)^{1/2} (u \times v)^2 v - \frac{1}{2n} [(2n)^{1/2} u^Av^A] \times [(2n)^{1/2} u^Bv^C] .
\]

(2.3)

With the help of Eqs. (3.3)–(3.5) in Ref. 4 and the definition of the zero component \(g^0\) of \(g\) therein, we have \(IT(\sqrt{2n} u^Av^A) = 2ng^0\) and hence

\[
IT(u^Av^A) = 2n[\langle gh \rangle^2 - g^2 h^2] .
\]

(2.4)

Combined with Eq. (2.1), we get

\[
\langle g,h \rangle = 2n \text{Tr} [\langle gh \rangle^2 - g^2 h^2] .
\]

(2.5)

Next we consider the properties of \(\text{Tr} g^2\). As defined in Eq. (2.2), \(\text{Tr}\) is cyclic invariant, hence...
\textbf{III. THE STRUCTURE OF THE THREE-QUARK COLOR ALGEBRA}

Let \( r = (123) \in S_3 \). Define the operator \( R \) acting on the color algebra by

\[ R(g, i) = r(g, i)r^{-1} = (rg^{-1}, r(i)), \]

i.e., \( R \) cyclically permutes the quark labels. Consider the following elements:

\begin{align*}
Q_1 &= (1 + R + R^2)(01), \\
Q_2 &= (1 + R + R^2)(23)(01), \\
Q_3 &= (1 + R + R^2)(012) + (021), \\
Q_4 &= (1 + R + R^2)(0123) + (0321), \\
Q_5 &= -3^{1/2}i(1 + R + R^2)(012) - (021), \\
Q_6 &= -3^{1/2}i(1 + R + R^2)(0123) - (0321).
\end{align*}

Straightforward computation with the help of a multiplication table constructed from Eq. (2.13) gives

\[ [Q_i, Q_j] = 0 \quad \forall i, j. \tag{3.2} \]

For each element \( u \) of the color algebra, we associate the operator \( L_u \) acting on the algebra defined by

\[ L_u v = [u, v]. \tag{3.3} \]

Since the Jacobi identity need not be true, we are not guaranteed that \( L_{(u, v)} = [L_u, L_v] \). However, it turns out that we have also

\[ [L_{Q_i}, L_{Q_j}] = 0 \quad \forall i, j. \tag{3.4} \]

Moreover, the following relations hold:

\begin{align*}
(n^2 - 2)L_{Q_1} - 2nL_{Q_2} + 4L_{Q_4} &= 0, \tag{3.5} \\
(n^2 - 2)L_{Q_2} + L_{Q_3} - nL_{Q_6} &= 0. \tag{3.6}
\end{align*}

Equations (3.5) and (3.6) imply that the following two elements commute with the whole algebra:

\begin{align*}
A_1 &= (n^2 - 2)Q_1 - 2(nQ_2 - 2Q_4), \tag{3.7} \\
A_2 &= (n^2 - 2)Q_2 + Q_3 - nQ_4. \tag{3.8}
\end{align*}

Using the fact that \( Q_1 \) annihilates the color-singlet state, one can show that \( Q_3 \), \( Q_2 \), \( Q_4 \) also annihilate the color-singlet state by multiplying \( Q_1 \) on the left with appropriate elements and projecting out the cyclically invariant part. Hence, both \( A_1 \) and \( A_2 \) annihilate the color-singlet state.

Equation (3.4) enables us to simultaneously diagonalize \( L_{Q_i} \) and characterize the elements of the color algebra by the eigenvalues \( \lambda_i \) of \( L_{Q_i} \) (only four of them are independent). In order to present the results, we need some definitions. The following definitions are made with hindsight.
Define the projection operators
\[ P_1 = 2 \times 3^{-1/2} \omega (1 + \omega^2 R + \omega R^2), \]
\[ P_2 = -2 \times 3^{-1/2} i \omega (1 + \omega R + \omega^2 R^2), \]
where \( \omega = \exp(i \varphi R) \), and the six elements
\[
\begin{align*}
    e_1 &= (012) + (021) - 2(12)(03), \\
    e_2 &= (0123) - (0312) + (0213) - (0321), \\
    e_3 &= (012) + (021) + (12)(03), \\
    e_4 &= (01) - (02), \\
    e_5 &= (012) - (021), \\
    e_6 &= (0123) - (0321) - (0213).
\end{align*}
\]

The simultaneous eigenvectors are expressed in terms of the two projection operators and the six elements.

Next we define some constants which arise as we solve the secular equations for eigenvalues:
\[ K_n = 3 + (4n^2 + 9)^{1/2}, \quad \overline{K}_n = 3 - (4n^2 + 9)^{1/2}, \]
\[ \alpha_n = \frac{4n}{K_n(K_n - \overline{K}_n)} \left( n - 3 \frac{K_n}{2} \right), \]
\[ \alpha_n = \frac{4n}{K_n(K_n - \overline{K}_n)} \left( n + 3 + \frac{K_n}{2} \right). \]

\( \alpha_n, \overline{\alpha}_n \) are defined by interchanging \( K_n \) and \( \overline{K}_n \) in Eq. (3.13).

Finally, define
\[
\begin{align*}
    v_3 &= \frac{2}{K_n(K_n - \overline{K}_n)} (n q_3 + \frac{K_n}{2} q_5), \\
    \overline{v}_3 &= \frac{-2}{\overline{K}_n(K_n - \overline{K}_n)} (n q_3 + \frac{\overline{K}_n}{2} q_5), \\
    q_n &= (n + 1)q_3 - 2q_2 + nq_4, \\
    q_n &= (n - 1)q_3 + 2q_2 - nq_4.
\end{align*}
\]

Table I gives the simultaneous eigenvectors of \( L_r \). The first column gives the symbols of the elements. The second column expresses them as linear combinations of the group elements \( \{ q, t \} \). The third column gives the eigenvalues \( \lambda, \overline{\lambda}, \mu, \overline{\mu}, \) \( \mu \) of \( v_3, \overline{v}_3, [1/6n(n + 2)]q_3, [-1/6n(n - 2)]q_2 \) respectively. It is easy to show by using the trace property \( \langle u | v, w \rangle = \langle [u, v, w] \rangle \) that any two eigenvectors in Table I are orthogonal unless the corresponding eigenvalues \( \lambda, \overline{\lambda}, \mu, \overline{\mu}, \lambda \) are of opposite sign. In particular, each element is orthogonal to itself. To obtain an orthonormal basis, we have only to combine each pair of opposite eigenvalues in the standard way. For example, define
\[ \tilde{v}_1 = 2^{1/2} \frac{v_3 + v_5}{\sqrt{\langle v_3, v_5 \rangle / 17}}, \quad \tilde{v}_2 = 2^{1/2} \frac{v_3 - v_5}{i\langle v_3, v_5 \rangle / 17}, \]
and the four elements \( v_3, \overline{v}_3, q_n, q_n \) are orthogonal to each other by straightforward computation. The inner products are computed by first constructing an inner product table using Eqs. (2.5) and (2.8).

The results are as follows:
\[
\begin{align*}
    \langle v_3, v_5 \rangle &= -4(n + 2)(n^2 - 1)\alpha_n/(K_n - \overline{K}_n)\overline{\alpha}_n, \\
    \langle \overline{v}_3, \overline{v}_5 \rangle &= 4(n + 2)(n^2 - 1)\alpha_n/(K_n - \overline{K}_n)\overline{\alpha}_n, \\
    \langle v_3, v_5 \rangle &= \frac{1}{2}\langle v_3, v_5 \rangle, \\
    \langle v_3, \overline{v}_5 \rangle &= \frac{1}{2}\langle \overline{v}_3, \overline{v}_5 \rangle, \\
    \langle x_1, x_2 \rangle &= \langle x_3, x_4 \rangle = 48n(n + 2)(n^2 - 1), \\
    \langle x_1, x_4 \rangle &= \langle x_2, x_3 \rangle = 48n(n - 2)(n^2 - 1), \\
    \langle q_n, q_n \rangle &= 24n^2(n + 2)(n^2 - 1), \\
    \langle q_n, q_n \rangle &= 24n^2(n - 2)(n^2 - 1).
\end{align*}
\]

As the notation suggests, \( q_n, v_3, \overline{v}_3, \) span a subalgebra which is isomorphic to \( u(2) \). We denote it by \( V_2 \). Similarly, \( q_n, \overline{v}_3, v_3 \) span the subalgebra \( V_1 \) which is also isomorphic to \( u(2) \). We write \( V = V_1 \oplus V_2 \). The remaining eight-dimensional subspace \( V_3 \) consists of \( V_3 \) spanned by \( x_1 \), and \( V_3 \) spanned by \( x_4 \). Instead of using eigenvalues \( \lambda, \overline{\lambda}, \mu, \overline{\mu}, \lambda \) to characterize \( x_1, x_4 \), we shall use the smaller one \( \lambda, \mu, \overline{\lambda}, \mu \), where \( \mu = \pm 1 \) specifies whether it is \( x_1 \) or \( x_4 \), and \( \mu = \pm 1 \) respectively. \( \lambda \) is still the eigenvalue of \( v_3 \) and we use \( \lambda \) to denote the sign \( \pm \) of \( \lambda = \pm \alpha_n/2 \).

Now we can write down the remaining outer products obtained by straightforward computation:
\[
\begin{align*}
    [v_3, (\lambda, \mu)] &= \lambda(\lambda, \mu), \\
    [\overline{v}_3, (\lambda, \mu)] &= \overline{\lambda}(\lambda, \mu), \\
    [v_3, (-\lambda, \mu)] &= 2i\mu \overline{\lambda}(\lambda, \mu), \\
    \overline{[v_3, (-\lambda, \mu)]} &= 2i\mu \lambda(\lambda, \mu), \\
    [v_3, (\lambda, -\mu)] &= 0, \\
    [\overline{v}_3, (\lambda, -\mu)] &= 0, \\
    [q_n, (\lambda, \mu)] &= 0, \\
    [q_n, (\lambda, -\mu)] &= 0, \\
    [q_n, (\lambda, \mu)] &= 0, \\
    [\overline{q}_n, (\lambda, \mu)] &= 0, \\
    [\overline{q}_n, (\lambda, -\mu)] &= 0.
\end{align*}
\]

where \( n_\delta = 6n(2 + 6n) \),
\[
\begin{align*}
    \langle (\lambda, \mu), (\lambda, -\mu), \rangle &= \delta_{\mu\delta} 12K_\delta (\lambda, \mu, \mu, \mu, \mu), \\
    \langle (\lambda, \mu), (\lambda, -\mu), \rangle &= 0, \\
    \langle \overline{\lambda}, \mu, (\lambda, -\mu), \rangle &= 0, \\
    \langle (\lambda, \mu), \overline{\lambda}, (\lambda, \mu), \rangle &= 0, \\
    \langle (\lambda, \mu), \overline{\lambda}, (\lambda, \mu), \rangle &= 0.
\end{align*}
\]

Equations (3.19)–(3.21) together with \( [v_3, \overline{v}_3] = 2\overline{v}_3, [\overline{v}_3, v_3] = 2v_3, \) etc., give us the complete multiplication table in terms of the new basis.

Now we consider the triple product \( \{ u, v, w \} \) de-
TABLE I. Simultaneous eigenvectors of \( L_\Omega \),

\[
\begin{array}{lll}
v_+ & \frac{2n}{3^{1/2}K_0(K_0 - K_3)} P_1 \left[ e_{5/4} + \frac{K_0}{6n} 3^{1/2} i (e_{4} - e_{0}) \right] & (1, 0, 0, 0) \\
v_- & \frac{2n}{3^{1/2}K_0(K_0 - K_3)} P_1 \left[ e_{5/4} - \frac{K_0}{6n} 3^{1/2} i (e_{4} - e_{0}) \right] & (-1, 0, 0, 0) \\
\bar{v}_+ & \frac{-2n}{3^{1/2}K_0(K_0 - K_3)} P_1 \left[ e_{5/4} - \frac{K_0}{6n} 3^{1/2} i (e_{4} - e_{0}) \right] & (0, 1, 0, 0) \\
\bar{v}_- & \frac{-2n}{3^{1/2}K_0(K_0 - K_3)} P_1 \left[ e_{5/4} - \frac{K_0}{6n} 3^{1/2} i (e_{4} - e_{0}) \right] & (0, -1, 0, 0) \\
x_{1+} & P_1 \left[ \frac{1}{2} (e_{4} + e_{2}) + \frac{1}{2} 3^{1/2} i (e_{5} + \frac{5}{2} e_{4} + \frac{1}{2} e_{0}) \right] & (-\alpha_n - \frac{1}{2}, -\bar{\alpha}_n - \frac{1}{2}, 0, 0) \\
x_{2+} & P_1 \left[ \frac{1}{2} (e_{4} - e_{2}) + \frac{1}{2} 3^{1/2} i (e_{5} - \frac{5}{2} e_{4} - \frac{1}{2} e_{0}) \right] & (\alpha_n + \frac{1}{2}, \bar{\alpha}_n + \frac{1}{2}, -1, 0) \\
x_{3+} & P_1 \left[ \frac{1}{2} (e_{4} - e_{2}) - \frac{1}{2} 3^{1/2} i (e_{5} - \frac{5}{2} e_{4} - \frac{1}{2} e_{0}) \right] & (-\alpha_n + \frac{1}{2}, \bar{\alpha}_n - \frac{1}{2}, 1, 0) \\
x_{4+} & P_1 \left[ \frac{1}{2} (e_{4} + e_{2}) - \frac{1}{2} 3^{1/2} i (e_{5} + \frac{5}{2} e_{4} + \frac{1}{2} e_{0}) \right] & (\alpha_n + \frac{1}{2}, \bar{\alpha}_n - \frac{1}{2}, 1, 0) \\
x_{1-} & P_1 \left[ \frac{1}{2} (e_{4} + e_{2}) - \frac{1}{2} 3^{1/2} i (e_{5} - \frac{5}{2} e_{4} - \frac{1}{2} e_{0}) \right] & (-\alpha_n - \frac{1}{2}, \bar{\alpha}_n + \frac{1}{2}, 0, 1) \\
x_{2-} & P_1 \left[ \frac{1}{2} (e_{4} - e_{2}) + \frac{1}{2} 3^{1/2} i (e_{5} + \frac{5}{2} e_{4} + \frac{1}{2} e_{0}) \right] & (\alpha_n + \frac{1}{2}, \bar{\alpha}_n + \frac{1}{2}, 0, 1) \\
x_{3-} & P_1 \left[ \frac{1}{2} (e_{4} + e_{2}) + \frac{1}{2} 3^{1/2} i (e_{5} + \frac{5}{2} e_{4} + \frac{1}{2} e_{0}) \right] & (-\alpha_n + \frac{1}{2}, \bar{\alpha}_n + \frac{1}{2}, 1, 0) \\
x_{4-} & P_1 \left[ \frac{1}{2} (e_{4} - e_{2}) + \frac{1}{2} 3^{1/2} i (e_{5} + \frac{5}{2} e_{4} + \frac{1}{2} e_{0}) \right] & (-\alpha_n - \frac{1}{2}, \bar{\alpha}_n - \frac{1}{2}, 1, 0) \\
\end{array}
\]

finned as

\[
\{u, v, w\} = [u, v, w] + [w, u, v] + [v, w, u].
\]

(3.22)

They appear naturally in the field equations and in the gauge transformations of various quantities.

Using the trace property, one can show that

\[
\langle x, \{u, v, w\}\rangle = -\langle w, \{x, u, v\}\rangle,
\]

(3.23)

hence \( \{u, v, w\} \) is totally antisymmetric in all its arguments.

Since \( V_1 \) is a Lie algebra, \( \{u, v, w\} = 0 \) unless \( u, v, w \in V_2 \). From Eqs. (3.19)–(3.21), it is easily seen that \( \{u, v, w\} = 0 \) unless \( u, v, w \in V_2 \) or \( u, v, w \in V_2 \). Moreover, \( \{u, v, w\} \in V_{28} \) if \( u, v, w \in V_{28} \). Since each \( V_{28} \) is four dimensional, we can choose an orthonormal basis \( \mathcal{J}_{16} \) such that

\[
\langle \mathcal{J}_{16}, \mathcal{J}_{16}, \mathcal{J}_{16}, \mathcal{J}_{16} \rangle = N_\phi \epsilon_{164},
\]

for some constant \( N_\phi \). Let

\[
\mathcal{J}_{16} = \frac{x_{16} + x_{24}}{21/2(x_{16}, x_{24})^{1/2}}, \quad \mathcal{J}_{28} = \frac{x_{16} - x_{24}}{21/2(x_{16}, x_{24})^{1/2}}.
\]

We find, after a simple calculation, that

\[
N_\phi = 3/(4 + 2\theta)(n^2 - 1),
\]

(3.24)

and we can write

\[
\langle \mathcal{J}_{16}, \mathcal{J}_{16}, \mathcal{J}_{16}, \mathcal{J}_{16} \rangle = N_\phi \epsilon_{164}\mathcal{J}_{16}.
\]

(3.25)

There remain the triple products \( \{u, v, w\} \) with exactly two of them belonging to \( V_1 \) or \( V_2 \) to be computed. The results are as follows:

\[
\{\lambda, \mu, \nu, (\lambda, -\mu, \nu, v, w)\} = 24\mathcal{J}_{16} \left[ (\alpha_{n0} - 1)\gamma_{n0} u_3 + \alpha_{n0} \gamma_{n0} \bar{v}_3 \right],
\]

(3.26)

\[
\{\lambda, \mu, \nu, (\lambda, -\mu, \nu, v_3)\} = -12\mathcal{J}_{16} \left[ (\alpha_{n0} - 1)\gamma_{n0} u_3 + \alpha_{n0} \gamma_{n0} \bar{v}_3 \right],
\]

(3.27)

\[
\{\lambda, \mu, \nu, (\lambda, -\mu, \nu, -\gamma_{n0} v_3)\} = 0,
\]

\[
\{\lambda, \mu, \nu, (\lambda, -\mu, \nu, \bar{v}_3)\} = 0,
\]

\[
\{\lambda, \mu, \nu, (\lambda, -\mu, \nu, \gamma_{n0} v_3)\} = 0,
\]

(3.28)

\[
\{\lambda, \mu, \nu, (\lambda, -\mu, \nu, -\gamma_{n0} \bar{v}_3)\} = 0.
\]

There exists another set of equations obtained by interchanging \( \gamma_{n0}, \gamma_{n0}^*, \gamma_{n0}, \bar{\gamma}_{n0}, \bar{\gamma}_{n0}^*, \bar{\gamma}_{n0} \), \( \mathcal{J}_{16}, \mathcal{J}_{28}, \mathcal{J}_{30} \), respectively. The case when two of the arguments in \( \{u, v, w\} \) belong to \( V_1 \) can be derived from Eqs. (3.26)–(3.28) by using Eq. (3.23).

Finally we compute the effective charges. The charge \( (0) \) can be written

\[
(01) = \frac{1}{2}[Q_1 + (1 - R^2)e_1].
\]

(3.29)

(02) and (03) are obtained by operating on Eq. (3.29) with \( R \) as defined in Eq. (3.1). Using Eqs.
(3.7), (3.8), (3.14), and (3.15), we find
\[
Q_1 = \frac{1}{n^2-1} A_1 - \frac{8}{n(n^2-2)(n^2-4)} A_2
+ \frac{1}{n(n+2)} q^+ + \frac{1}{n(n-2)} q^-.
\] (3.30)

From Table I, we get
\[
P_1 e_4 = -i 2^{-1} \times 3^{-1/2} (x_1, x_{1+}, x_{2+}, x_{3+})
+ K e_5 + K e_6,
\] (3.31)
\[
P_2 e_4 = i 2^{-1} \times 3^{-1/2} (x_{2+}, x_{2+}, x_{4+}, x_{4+})
+ K e_6 - K e_5.
\] (3.32)

Finally, from Eqs. (3.9) and (3.10), we find
\[
1 - R^2 = i 2^{-1} \times 3^{-1/2} (1 - \omega) (P_1 + \omega P_2)
= -\frac{1}{2} (\omega P_1 + \omega P_2).
\] (3.33)

Hence the effective charge \(Q_{1, eff}\) is
\[
Q_{1, eff} = \frac{1}{2n} \left( \frac{1}{n+2} q^+ + \frac{1}{n-2} q^- \right)
- \frac{1}{3} \omega P_1 e_4 + \omega P_2 e_4,
\] (3.34)

with \(P_1 e_4, P_2 e_4\) given in Eqs. (3.31) and (3.32).

The other effective charges \(Q_{2, eff}\) and \(Q_{3, eff}\) are obtained by operating on Eq. (3.34) with \(R\).

To conclude this section, we consider briefly the case \(n=2\), which is special because for \(su(2)\), \(d^{ABC}=0\). First we note that for Hermitian operators \(u^A\), \(\text{Tr}(u^A u^B)\) is positive definite. In particular, the inner products computed in Eqs. (3.16)–(3.18) have to be positive which indeed is the case when \(n \geq 3\). For \(n=2\), some of them vanish, which shows that the corresponding elements vanish as a result of putting \(d^{ABC}=0\). Hence we see that for \(n=2\), \(x_{1+}, \bar{v}_1, \bar{v}_2\) all vanish. Moreover, \(q_-=A_2\) and we have only three independent combinations of \(Q_1, Q_2, Q_2, Q_4\). It is easy to show that the combination \(2Q_1 - Q_2\) also vanishes and accounts for the degeneracy \(q_- = A_2\). Summarizing, we have a nine-dimensional algebra spanned by \(x_{1+}, v_1, v_2, v_3, q_+, A_1\) with \(A_1\) commuting with the whole algebra. The remaining eight-dimensional subalgebra is not a Lie algebra as one may check that Eqs. (3.26)–(3.28) are nontrivial.

IV. DISCUSSION AND CONCLUSION

As we have seen, the three-quark color algebra with the outer product \([u, v]\) \(\mathcal{A} = \mathcal{A}^{ABC}(u^A v^B + v^C u^B)\) decomposes, aside from the two Abelian elements \(A_1, A_2\), which annihilate the color–singlet state, into a direct sum of two vector spaces \(V_1 \oplus V_2\) satisfying the relations \([V_1, V_1] = V_1\), \([V_1, V_2] = V_2\), \([V_2, V_2] = V_1\). \(V_1\) is a Lie algebra isomorphic to \(u(2) \oplus u(2)\). The integration constant \(K_{11}\) defined in Ref. 3 can be computed with the help of Eqs. (3.16)–(3.18) and (3.31)–(3.34). We find
\[
(Q_{1, eff}, Q_{1, eff}) = \frac{3}{2}(n^2 + 8)(n^2 - 1),
\] (4.1)
so that
\[
K_{11} = \frac{3}{4n(n^2 + 8)}.
\] (4.2)

Hence in the three-quark case, there seems to be no problem with the theory proposed by Adler in Ref. 3.

The method I used in deriving the results presented in Sec. III is not very useful in treating the case of many quarks. The dimension of the color algebra increases so fast that even if one can guess at some correct results concerning the structure of the color algebra it will be very difficult to verify them by straightforward computation. Whether the theory of Ref. 3 needs modification when more than three static sources are present remains to be seen.

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6The case of \(n=2\) requires special consideration and is discussed at the end of this section.