科目：線性代數(1002)　　校系所組：中大數學系甲組、乙組
清大數學系純粹數學組、應用數學組

(30%) 1. For each of the following functions, show that it is a linear transformation and determine whether it is invertible. If it is invertible, find an explicit formula for its inverse.

(15%) (a) $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = \int_x^{x+1} f(t)dt$, where $P_3(\mathbb{R})$ denotes the space consisting of polynomials with real coefficients of degree less than or equal to 3.

(15%) (b) $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ defined by $T(A) = A + \mathrm{tr}(A)I_n$, where $M_{n\times n}(\mathbb{R})$ denotes the space consisting of $n \times n$ matrices with real entries, $\mathrm{tr}(A)$ denotes the trace of A, and I_n denotes the identity matrix.

(15%) 2. Let V be a finite dimensional inner product space over the real numbers and let $T : V \to V$ be a linear operator on V. Define $\det(T) = \det([T]_\beta)$, where $[T]_\beta$ denotes the matrix representation of T in the ordered basis β.

(5%) (a) Show that $\det(T)$ is independent of the choice of the ordered basis β.

(10%) (b) Suppose that $||T(v)|| = ||v||$ for all $v \in V$. Prove that $\det(T) = \pm 1$.

(10%) 3. Let V be the space of polynomials with real coefficients with inner product $< f(x), g(x) >= \int_{-1}^{1} f(x)g(x)dx$. Let W be the subspace consisting of even polynomials. Prove or disprove that $V = W \oplus W^\perp$, where $W^\perp$ is the orthogonal complement of W.

(15%) 4. For $B \in M_{m\times m}(\mathbb{R})$, define the linear operator T on the matrix space $M_{m\times n}(\mathbb{R})$ by $T(A) = BA$.

(5%) (a) Prove that T is is invertible if and only if B is invertible.

(5%) (b) Prove that T and B have the same eigenvalues.

(5%) (c) Prove that T is diagonalizable if and only if B is diagonalizable.

(15%) 5. Let r be a positive real number and let m be a positive integer. Let $A = [a_{ij}]$ be an $m \times m$ matrix given by

$$a_{ij} = \begin{cases} r^{i-1} & \text{if } i + j = 1 + m, \\ 0 & \text{otherwise.} \end{cases}$$

(5%) (a) Show that $\pm(\sqrt{r})^{m-1}$ are the only eigenvalues of A.

(10%) (b) Find the minimal polynomial of A and evaluate A^{2008}.

(15%) 6. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a nonzero linear operator on V. Denote the minimal polynomial of T by $f(x)$. Prove or disprove that $V = N(T) \oplus R(T)$ if and only if $x^2 \nmid f(x)$.