國立清華大學100學年度碩士班入學考試試題

系所班組別：數學系碩士班純粹數學組

考試科目（代碼）：高等微積分(0101)

　　*請在【答案卷、卡】作答

Advanced Calculus Written Exam

1. (18%) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n^2$ converges. Are the following series convergent? Prove your answer or provide a counterexample.

   (a) $\sum_{n=1}^{\infty} \sin(a_n)$　　　(b) $\sum_{n=1}^{\infty} \frac{a_n}{n}$

2. (18%) Let A be an open subset of $\mathbb{R}^n$. Prove or disprove, by giving a counterexample, the following statements.

   (a) $\text{int}(\overline{A}) = A$.

   (b) $A \cap \overline{B} \subset \overline{A \cap B}$ for any $B \subset \mathbb{R}^n$.

   (Here "int" means interior, upper bar means closure.)

3. (12%) Suppose $a_n \in \mathbb{R}$ and $|a_n| \le \frac{n^2}{2^n}$ for every positive integer n. Let

$$f(x) = \sum_{n=1}^{\infty} a_n x^n, \quad f_k(x) = f\left(x + \frac{1}{k}\right).$$

   Show that f_k converges uniformly to f on $[-1, 1]$ as $k \to \infty$.

4. (12%) Let $f(x, y) = \sin\left(\frac{1}{|x-y|^2}\right)$ when $x \ne y$ and let $f(x, x) = 0$ for any x. Is this function Riemann integrable on $[0, 1] \times [0, 1]$? Prove your answer.

5. (12%) Define $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ by $f(x) = \frac{x}{\|x\|}$. Calculate the derivative $Df(x)$ for f at $x \in \mathbb{R}^n \setminus \{0\}$.

6. (14%) Show that the solution set for the system

$$\begin{cases} x^2 + y^2 + z^2 &= 1 \\ x^3 + y^3 + z^3 &= 0 \end{cases}$$

   is a smooth curve (i.e. a curve of class C^{∞}) in $\mathbb{R}^3$.

7. (14%) Evaluate the double integral

$$\iint_E e^{x^2 + 2xy + 5y^2} \, dA$$

   where E is the ellipse $\{(x, y) \in \mathbb{R}^2 : x^2 + 2xy + 5y^2 \le 1\}$.