國立清華大學 100 學年度碩士班入學考試試題

系所班組別：資訊系統與應用研究所 甲組

考試科目（代碼）：機率論 (2302)

1. (8%) In a lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select

   (a) (4%) 3, 4, 5, 6

   (b) (4%) 4, 4, 6, 6

2. (8%) Suppose there are four urns, where urn U_1 contains 3 red balls, urn U_2 contains 2 black balls, urn U_3 contains 2 red balls and 2 black balls, and urn U_4 contains 1 red ball and 3 black balls. The probabilities of selecting U_1, U_2, U_3, or U_4 are 1/2, 1/4, 1/8, and 1/8, respectively. An urn is selected and a ball is then drawn at random.

   (a) (4%) Find the probability of drawing a red ball.

   (b) (4%) Find the conditional probability that U_4 had been selected, given that a red ball is selected.

3. (16%) Let X equal the number of flips of a fair coin that are required to observe the same face on consecutive flips.

   (a) (4%) Find the probability mass function of X.

   (b) (4%) Find the value of the mean of X.

   (c) (4%) Find the value of the variance of X.

   (d) (4%) Find the value of $P(X \geq 4)$.

4. (10%) Suppose that variables $X_1, \ldots, X_n$ form a random sample of size n from a uniform distribution on the interval (0,1) and the random variables Y_1 and Y_n are defined as $Y_1 = \min\{X_1, \ldots, X_n\}$ and $Y_n = \max\{X_1, \ldots, X_n\}$.

(a) (5%) Determine the value of $\Pr(Y_1 \le 0.2 \text{ and } Y_n \le 0.7)$.

(b) (5%) Determine the probability that the interval from Y_1 to Y_n will not contain the point 1/6.

5. (15%) Suppose that X and Y have a continuous joint distribution for which the joint probability density function is as follows:

$$f(x,y) = \begin{cases} x+y & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0 & \text{otherwise.} \end{cases}$$

Find the expectation $E(Y|X)$ and the variance $\mathrm{Var}(Y|X)$.

6. (10%) Suppose that 25 percent of the photos in a photo collection contain skies. For each photo, its blue hue is measured by taking the average of the blue channel over all pixels. For photos that contain skies, the values of the blue hue X will be normally distributed with a mean of 200 and a variance of 20. For photos that do not contain skies, the blue hue X will be normally distributed with a mean of 100 and a variance of 20. Suppose that a photo is selected at random from the collection and its blue hue X is measured.

(a) (5%) Determine the conditional probability that the photo contains a sky given that $X=x$.

(b) (5%) For what values of x is the conditional probability in (a) greater than 0.5?

7. (15%) An email account receives 1 email every 10 minutes in average. Assume the email arrival for this account is a Poisson process. Let a random variable X denote the total number of emails received in one hour by this account.

(a) (5%) Write down the probability distribution for the random variable X.

(b) (5%) What is the probability that this email account receives less than 2 emails in one hour?

(c) (5%) Let a random variable Y denote the time (in minutes) between two emails received by this account in sequence. Write down the probability distribution function for Y.

8. (18%) Let T be the time between emissions of particles by a radio-active atom. It is assumed that T is a random variable with an exponential distribution. Its probability density function is given as follows:

$$f(t) = \begin{cases} \lambda\, e^{-\lambda t} & t \geq 0 \\ 0 & elsewhere \end{cases}$$

where λ is a positive constant.

(a) (6%) Derive the mean and variance of the random variable T.

(b) (6%) Derive the cumulative distribution function for the random variable T.

(c) (6%) What are the probabilities $P(T = 1/\lambda)$ and $P(T > 2/\lambda)$? Show your calculation.