台灣聯合大學系統99學年度碩士班考試命題紙

科目：工程數學 C(5005)

校系所組：中央大學電機工程學系(電子組、系統與生醫組)
交通大學電子研究所(甲組、乙組)
交通大學電控工程研究所(甲組、丙組)
交通大學電信工程研究所（乙組）
清華大學電機工程學系(甲組)
清華大學光電工程研究所
清華大學電子工程研究所
清華大學工程與系統科學系（丁組）

一. (5%)

(一) (2%) Solve $y' = y^2, y(0) = 2$. Call the solution y_c.

(二) (2%) Solve $y' = y^2 - 1, y(0) = 1$. Call the solution y_p.

(三) (1%) Does $y_c + y_p$ solve $y' = y^2 - 1, y(0) = 3$? Explain.

二. (8%) One solution of the equation $y'' + p(t)y' + q(t)y = 0$ is $(1+t)^2$, and the Wronskian of any two solutions is constant. Find the general solution of $y'' + p(t)y' + q(t)y = 1 + t$.

三. (5%) Three solutions of a 2nd-order linear equation $L(y) = g(t)$ are

$$\psi_1 = 2e^{t^2} + e^t, \psi_2 = te^{t^2} + e^t \text{ and } \psi_3 = (1+t)e^{t^2} + e^t.$$

Find the solution of the initial problem $L(y) = g(t); y(0) = 3, y'(0) = 0$.

四. (8%) Let y be a real function of x. Find two linearly independent Frobenius solutions of the following differential equation at $x = 0$:

$$2x^2 y'' + x(x-3)y' + 3y = 0$$

五. (8%) Let x_1 and x_2 be two real functions of t. Solve x_1 and x_2 for the following system of differential equations

$$\begin{cases} x_1' = 4x_1 - x_2 \\ x_2' = x_1 + 2x_2 \end{cases}, x_1(1) = 5, x_2(1) = 3$$

六. (7%) Given the initial value problem, $x'' + 4x' + 13x = f(t);\ x(0) = x'(0) = 0$

(一) (3%) Express $x(t)$ in terms of $f(t)$ and convolution.

(二) (4%) Solve $x(t)$ for $f(t) = u(t) - u(t-1)$, where $u(t)$ denotes the unit step (or Heaviside Step) function.

注意：背面有試題

七. (9%) $f(t)=\begin{cases}1, & 0<t<5\\ 0, & 5<t<10\end{cases}$ *with* $f(t+10)=f(t)$ is a piecewise continuous and periodic function that satisfies $f(t)=\dfrac{\left|f(t^{+})+f(t^{-})\right|}{2}$, where $f(t^{+})$ and $f(t^{-})$ are the right-hand and left-hand limits of $f(t)$ at each discontinuity.

(一) (3%) Find the Fourier series of $f(t)$.

(二) (3%) Let $f(t)$ be defined for $t\ge 0$; find its Laplace transform $F(s)$ for $s>0$.

(三) (3%) Find a particular solution for $x''+16x=f(t)$.

八. (9%) Consider the following partial differential equation: $\dfrac{\partial^2 u}{\partial t^2}=4\dfrac{\partial^2 u}{\partial x^2}$

For the moment let's assume that u is in the form of $u=A\sin\lambda x\cos\omega t$, where A, λ, and ω are non-zero constants.

(一) (3%) What is the relationship between λ and ω?

(二) (3%) Assuming that u must satisfy the condition $u(x=0,t)=u(x=8,t)=0$, what is λ? Notice that λ might not be single valued.

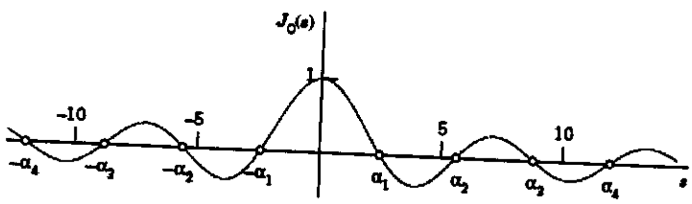
(三) (3%) What is the lowest possible frequency of u?

九. (4%) Given the equation for the circular membrane vibration: $\dfrac{\partial^2 u}{\partial t^2}=c^2\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}\right)$

with the boundary condition: $u(r=20,t)=0$ and c is a constant, the solution is in the form of

$$u=\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}W_nT_n=\sum_{n=1}^{\infty}(A_n\cos ck_nt+B_n\sin ck_nt)J_0(k_nr)$$

where J_0 is the Bessel function of the first kind. Find the nodal line (curves on the membrane that do not move) for $n=2$. Note that J_0 has infinitely many positive zeros α_1, α_2, α_3 ... as shown in the following figure.



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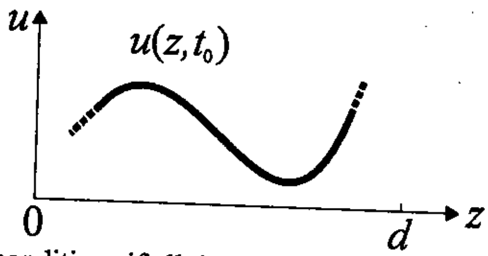
十. (12%) Consider a bulk of silicon with dimension $l \times w \times d$, where the thickness d (along the z-axis) is much smaller than the length l and width w. The diffusion and annealing of dopant ions is governed by the diffusion equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial z^2}, \ (t > 0,\ 0 < z < d)$$

where $u(z,t)$ means the dopant ion density and α^2 represents the diffusivity.

(一) (6%) Let the dopant ion density profile somewhere inside the silicon at a certain instant t_0, i.e. $u(z,t_0)$, look like as shown in the figure. Without detailed calculation, please roughly draw the new dopant ion density profile corresponding to a very short time Δt later, i.e. $u(z,t_0+\Delta t)$.



(二) (3%) Write down the boundary conditions if all the dopant ions cannot escape from the bulk of silicon.

(三) (3%) Without detailed calculation, can you draw the steady-state dopant ion density profile $u(z,t\to\infty)$ by using $u(z,t_0)$ as shown and the boundary conditions specified in part (二)?

十一. (13%) For a 3x3 matrix A, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$

(一) (6%) Please find its eigenvalues and corresponding eigenvectors.

(二) (7%) Assume the 3 eigenvalues are in the order of $\lambda_1 \le \lambda_2 \le \lambda_3$. Starting from the eigenvector corresponding to λ_1, please find the corresponding orthonormal basis.

十二. (12%) Define the space P_n as the set of all polynomials of degree less than n. Let L be the operator on P_3 and

$$L(p(x)) = \alpha + x\frac{d}{dx}p(x) + \frac{d^2}{dx^2}p(x)$$

(一) (3%) Find the matrix **A** representing L with respect to $[1, x, x^2]$.

(二) (3%) Find the matrix **B** representing L with respect to $[1, x, 1+x^2]$.

(三) (3%) Find the condition of α such that **A** and **B** are similar matrices.

(四) (3%) If $p(x) = a_0 + a_1x + a_2x^2$, calculate $L^n(p(x))$ given the condition of α in (三).